

CRYSTAL BASES FOR THE QUANTUM QUEER SUPERALGEBRA AND SEMISTANDARD DECOMPOSITION TABLEAUX

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ABSTRACT. In this paper, we give an explicit combinatorial realization of the crystal $B(\lambda)$ for an irreducible highest weight $U_q(\mathfrak{q}(n))$ -module $V(\lambda)$ in terms of semistandard decomposition tableaux. We present an insertion scheme for semistandard decomposition tableaux and give algorithms of decomposing the tensor product of $\mathfrak{q}(n)$ -crystals. Consequently, we obtain explicit combinatorial descriptions of the shifted Littlewood-Richardson coefficients.

INTRODUCTION

The queer Lie superalgebra $\mathfrak{q}(n)$ is the second super analogue of the general linear Lie algebra $\mathfrak{gl}(n)$ and is one of the most interesting algebraic structures studied both by mathematicians and physicists. It has been known since its inception that the representation theory of $\mathfrak{q}(n)$ is rather complicated. This is partly due to the fact that every Cartan subalgebra of $\mathfrak{q}(n)$ is noncommutative, and, as a result, the highest weight space of a highest weight $\mathfrak{q}(n)$ -module has a Clifford module structure. The combinatorial structure of the finite-dimensional $\mathfrak{q}(n)$ -modules is especially interesting. The *tensor representations* of $\mathfrak{q}(n)$; i.e., those that appear as subrepresentations of tensor powers of the vector representation, are involved in the queer analogue of the celebrated *Schur-Weyl duality*, often referred to as the *Schur-Weyl-Sergeev duality*. This duality was obtained in [19] for $U(\mathfrak{q}(n))$ and in [17] for $U_q(\mathfrak{q}(n))$. The isomorphism classes of

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tensor representations of $\mathfrak{q}(n)$ are parametrized by the set Λ^+ of strict partitions of n . Another important property of the tensor representations is that their characters are multiples of the so-called *Schur's Q -functions* [19].

The combinatorial description of the tensor modules over $U_q(\mathfrak{q}(n))$ can best be understood in the language of *crystal bases*. Originally introduced in [11, 12] for integrable modules over quantum groups associated with symmetrizable Kac-Moody algebras, the crystal basis theory is considered nowadays as one of the most prominent discoveries in the combinatorial representation theory. Some of the significant features of the crystal bases include: an extremely nice behavior with respect to tensor products and important connections with combinatorics of Young tableaux and Young walls (see, for instance, [1, 8, 10, 14, 15, 16]).

The numerous combinatorial applications of the queer Lie superalgebra together with the fact that the category of tensor representations is semisimple raise a natural question - is there a crystal basis theory for this category? The answer to this question is affirmative and the solution has been recently obtained in two steps. First, the highest weight module theory for $U_q(\mathfrak{q}(n))$ was developed in [5]. Second, the crystal basis theory for the category $\mathcal{O}_{\text{int}}^{\geq 0}$ of tensor representations over $U_q(\mathfrak{q}(n))$ was established in [3]. As one can expect, due to the queer nature of $\mathfrak{q}(n)$, the definition of a crystal basis for modules in $\mathcal{O}_{\text{int}}^{\geq 0}$ is different from the one used for all other categories of representations studied so far. A *crystal basis* for a $U_q(\mathfrak{q}(n))$ -module M in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ is defined to be a triple $(L, B, (l_b)_{b \in B})$, where the crystal lattice L is a free $\mathbf{C}[[q]]$ -submodule of M , B is a finite $\mathfrak{gl}(n)$ -crystal, $(l_b)_{b \in B}$ is a family of nonzero vector spaces such that $L/qL = \bigoplus_{b \in B} l_b$, with a set of compatibility conditions for the action of the Kashiwara operators. The main result in [3] is the existence and uniqueness theorem for a crystal basis of any module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. The crystal corresponding to an irreducible highest weight module $V(\lambda)$ is denoted by $B(\lambda)$.

Once a crystal basis theory for the category $\mathcal{O}_{\text{int}}^{\geq 0}$ is established, the next task is to look at the following two problems:

- (a) to find an explicit combinatorial realization of the crystal $B(\lambda)$,
- (b) to establish a Littlewood-Richardson rule for the tensor product of crystals $B(\lambda) \otimes B(\mu)$.

The purpose of this paper is to solve these problems. A class of combinatorial objects that describe the tensor representations of $\mathfrak{q}(n)$ has been known for more than thirty years - the *shifted semistandard Young tableaux*. These objects have been extensively studied by Sagan, Stembridge, Worley, and others, leading to important and deep results (in particular, the shifted Littlewood-Richardson rule) [18, 21, 22]. Unfortunately, the set of shifted semistandard Young tableaux of fixed shape does not have a natural crystal structure, and for this reason we use another setting.

Our approach is based on the notion of semistandard decomposition tableaux, which was first introduced by Serrano [20]. For our purpose, we slightly change the definition used in [20]. In our setting, a *hook word* is a word $u = u_1 \cdots u_N$ for which $u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_N$ for some k . Then, a *semistandard decomposition tableau* is a filling T of a shifted shape $\lambda = (\lambda_1, \dots, \lambda_n)$ with elements of $\{1, 2, \dots, n\}$ such that

(i) the word v_i formed by reading the i -th row from left to right is a hook word of length λ_i ,

(ii) v_i is a hook subword of maximal length in $v_{i+1}v_i$ for $1 \leq i \leq r-1$, where r is the number of nonzero λ_i 's.

Our first main result is that the set $\mathbf{B}(\lambda)$ of all semistandard decomposition tableaux of shifted shape λ has a crystal structure and is isomorphic to $B(\lambda)$. This combinatorial realization of crystals and the properties of lowest weight vectors enable us to decompose the tensor product $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ into a disjoint union of connected components as follows:

$$(0.1) \quad \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \simeq \bigoplus_{\nu \in \Lambda^+} \mathbf{B}(\nu)^{\oplus f_{\lambda, \mu}^{\nu}},$$

where $f_{\lambda, \mu}^{\nu} = |LR_{\lambda, \mu}^{\nu}|$ and

$$LR_{\lambda, \mu}^{\nu} = \{u = u_1 \cdots u_N \in \mathbf{B}(\lambda) ; \begin{array}{l} \text{(a) } \text{wt}(u) = w_0(\nu - \mu) \text{ and} \\ \text{(b) } \mu + \epsilon_{n-u_N+1} + \cdots + \epsilon_{n-u_k+1} \in \Lambda^+ \text{ for all } 1 \leq k \leq N \end{array} \}.$$

We call $f_{\lambda, \mu}^{\nu}$ the *shifted Littlewood-Richardson coefficient*.

As seen in (0.1), the connected component containing $T \otimes T' \in \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ is isomorphic to $\mathbf{B}(\nu)$ for some ν . In order to find ν and the element S of $\mathbf{B}(\nu)$ corresponding to $T \otimes T'$ explicitly, we consider the *insertion scheme* for semistandard decomposition tableaux. Namely, for a semistandard decomposition tableau T and a letter x , we define the filling $T \leftarrow x$ and prove that $T \leftarrow x$ is a semistandard decomposition tableau and that $S = T \leftarrow x$ and $\nu = \text{sh}(T \leftarrow x)$ the shape of $T \leftarrow x$. From here one easily defines $T \leftarrow T'$ for semistandard decomposition tableaux T and T' . Our insertion scheme is analogous to the one introduced in [20] and can be considered as a variation of those used for shifted tableaux by Fomin, Haiman, Sagan, and Worley, [2, 6, 18, 22]. It turns out that there exists a crystal isomorphism between the connected component containing $T \otimes T'$ and the crystal $\mathbf{B}(\text{sh}(T \leftarrow T'))$ sending $T \otimes T'$ to $T \leftarrow T'$. A crucial part of the proof is a queer version of the *Knuth relation*, which is a crystal isomorphism between certain sets of four-letter words. Using this insertion scheme and the properties of lowest weight vectors, we obtain the following decomposition:

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \simeq \bigoplus_{\substack{T \in \mathbf{B}(\lambda); \\ T \leftarrow L^{\mu} = L^{\nu} \text{ for some } \nu}} \mathbf{B}(\text{sh}(T \leftarrow L^{\mu})).$$

Here, L^{ν} denotes a unique lowest weight vector in $\mathbf{B}(\nu)$.

Finally, we introduce the notion of *recording tableaux* of the insertion scheme. The recording tableaux characterize the connected components in $\mathbf{B}^{\otimes N}$ or $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$. That is, any two elements are in the same connected component if and only if they have the same recording tableau. For the insertion $P(u) = (\cdots((u_1 \leftarrow u_2) \leftarrow u_3) \cdots) \leftarrow u_N$ of the word $u = u_1 u_2 \cdots u_N \in \mathbf{B}^{\otimes N}$, the recording tableau $Q(u)$ is defined to be the *standard shifted tableau* of the same shape as $P(u)$ which records the newly added boxes with $1, 2, \dots, N$. One can show that to each standard shifted tableau there is a unique lowest weight vector in $\mathbf{B}^{\otimes N}$. Hence the multiplicity of $\mathbf{B}(\lambda)$ in $\mathbf{B}^{\otimes N}$ is equal to the number of standard shifted tableaux of shape λ , which is denoted by f^λ .

On the other hand, when $u_1 u_2 \cdots u_N$ is the reading word of T , we define the insertion $T \rightarrow T'$ by $u_1 \leftarrow (u_2 \leftarrow \cdots \leftarrow (u_{N-1} \leftarrow (u_N \leftarrow T')) \cdots)$. Then the corresponding recording tableau $Q = Q(T \rightarrow T')$ is defined by the following conditions:

- (a) Q is a standard shifted tableau of shape ν/μ , where $\nu = \text{sh}(T \rightarrow T')$ and $\mu = \text{sh}(T')$,
- (b) $(n - r_{|\lambda|} + 1) \otimes (n - r_{|\lambda|-1} + 1) \otimes \cdots \otimes (n - r_1 + 1)$ is a semistandard decomposition tableau of shifted shape λ , where $\lambda = \text{sh}(T)$ and r_k denotes the row of the entry k in Q .

Such a tableau Q is called the *shifted Littlewood-Richardson tableau of shape ν/μ and type λ* and we show that there is a 1-1 correspondence between the set of shifted Littlewood-Richardson tableaux of shape ν/μ and type λ and the set of lowest weight vectors of weight $w_0(\nu)$ in $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$. It follows that the number of the shifted Littlewood-Richardson tableaux of shape ν/μ and type λ gives the shifted Littlewood-Richardson coefficient $f_{\lambda, \mu}^\nu$.

This paper is organized as follows. In Section 1, we collect some important definitions and facts on the crystal basis theory for the $U_q(\mathfrak{q}(n))$ -modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. In Section 2, we introduce the notion of semistandard decomposition tableaux and prove that $\mathbf{B}(\lambda) \simeq B(\lambda)$. Furthermore, we give the shifted Littlewood-Richardson rule for $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$. Section 3 is devoted to the insertion scheme and its properties, while in the last section we present the notion of the recording tableaux and give another description of the shifted Littlewood-Richardson coefficients.

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1. CRYSTAL BASES FOR THE QUANTUM QUEER SUPERALGEBRA

1.1. The quantum queer superalgebra. Let $\mathbf{F} = \mathbf{C}((q))$ be the field of formal Laurent series in an indeterminate q and let $\mathbf{A} = \mathbf{C}[[q]]$ be the subring of \mathbf{F} consisting

of formal power series in q . For $k \in \mathbf{Z}_{\geq 0}$, we define

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [0]! = 1, \quad [k]! = [k][k-1] \cdots [2][1].$$

For an integer $n \geq 2$, let $P^\vee = \mathbf{Z}k_1 \oplus \cdots \oplus \mathbf{Z}k_n$ be a free abelian group of rank n and let $\mathfrak{h}_0^\vee = \mathbf{C} \otimes_{\mathbf{Z}} P^\vee$ be the *even part of the Cartan subalgebra*. Define the linear functionals $\epsilon_i \in \mathfrak{h}_0^*$ by $\epsilon_i(k_j) = \delta_{ij}$ ($i, j = 1, \dots, n$) and set $P = \mathbf{Z}\epsilon_1 \oplus \cdots \oplus \mathbf{Z}\epsilon_n$. We denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ the *simple roots* and by $h_i = k_i - k_{i+1}$ the *simple coroots*.

Definition 1.1. *The quantum queer superalgebra $U_q(\mathfrak{q}(n))$ is the superalgebra over \mathbf{F} with 1 generated by the symbols $e_i, f_i, e_{\bar{i}}, f_{\bar{i}}$ ($i = 1, \dots, n-1$), q^h ($h \in P^\vee$), $k_{\bar{j}}$ ($j = 1, \dots, n$) with the following defining relations.*

$$\begin{aligned}
(1.1) \quad & q^0 = 1, \quad q^{h_1} q^{h_2} = q^{h_1+h_2} \quad (h_1, h_2 \in P^\vee), \\
& q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \quad (h \in P^\vee), \\
& q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee), \\
& q^h k_{\bar{j}} = k_{\bar{j}} q^h, \\
& e_i f_j - f_j e_i = \delta_{ij} \frac{q^{k_i - k_{i+1}} - q^{-k_i + k_{i+1}}}{q - q^{-1}}, \\
& e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{if } |i - j| > 1, \\
& e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad \text{if } |i - j| = 1, \\
& f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad \text{if } |i - j| = 1, \\
& k_{\bar{i}}^2 = \frac{q^{2k_i} - q^{-2k_i}}{q^2 - q^{-2}}, \\
& k_{\bar{i}} k_{\bar{j}} + k_{\bar{j}} k_{\bar{i}} = 0 \quad \text{if } i \neq j, \\
& k_{\bar{i}} e_i - q e_i k_{\bar{i}} = e_{\bar{i}} q^{-k_i}, \quad q k_{\bar{i}} e_{i-1} - e_{i-1} k_{\bar{i}} = -q^{-k_i} e_{\bar{i}-1}, \\
& k_{\bar{i}} e_j - e_j k_{\bar{i}} = 0 \quad \text{if } j \neq i, i-1, \\
& k_{\bar{i}} f_i - q f_i k_{\bar{i}} = -f_{\bar{i}} q^{k_i}, \quad q k_{\bar{i}} f_{i-1} - f_{i-1} k_{\bar{i}} = q^{k_i} f_{\bar{i}-1}, \\
& k_{\bar{i}} f_j - f_j k_{\bar{i}} = 0 \quad \text{if } j \neq i, i-1, \\
& e_i f_{\bar{j}} - f_{\bar{j}} e_i = \delta_{ij} (k_{\bar{i}} q^{-k_{i+1}} - k_{\bar{i+1}} q^{-k_i}), \\
& e_{\bar{i}} f_j - f_j e_{\bar{i}} = \delta_{ij} (k_{\bar{i}} q^{k_{i+1}} - k_{\bar{i+1}} q^{k_i}), \\
& e_i e_{\bar{i}} - e_{\bar{i}} e_i = f_i f_{\bar{i}} - f_{\bar{i}} f_i = 0, \\
& e_i e_{i+1} - q e_{i+1} e_i = e_{\bar{i}} e_{\bar{i+1}} + q e_{\bar{i+1}} e_{\bar{i}}, \\
& q f_{i+1} f_i - f_i f_{i+1} = f_{\bar{i}} f_{\bar{i+1}} + q f_{\bar{i+1}} f_{\bar{i}}, \\
& e_i^2 e_{\bar{j}} - (q + q^{-1}) e_i e_{\bar{j}} e_i + e_{\bar{j}} e_i^2 = 0 \quad \text{if } |i - j| = 1,
\end{aligned}$$

$$f_i^2 f_{\bar{j}} - (q + q^{-1}) f_i f_{\bar{j}} f_i + f_{\bar{j}} f_i^2 = 0 \quad \text{if } |i - j| = 1.$$

The generators e_i, f_i ($i = 1, \dots, n-1$), q^h ($h \in P^\vee$) are regarded as *even* and $e_{\bar{i}}, f_{\bar{i}}$ ($i = 1, \dots, n-1$), $k_{\bar{j}}$ ($j = 1, \dots, n$) are *odd*. From the defining relations, it is easy to see that the even generators together with $k_{\bar{1}}$ generate the whole algebra $U_q(\mathfrak{q}(n))$.

The superalgebra $U_q(\mathfrak{q}(n))$ is a bialgebra with the comultiplication $\Delta: U_q(\mathfrak{q}(n)) \rightarrow U_q(\mathfrak{q}(n)) \otimes U_q(\mathfrak{q}(n))$ defined by

$$(1.2) \quad \begin{aligned} \Delta(q^h) &= q^h \otimes q^h \quad \text{for } h \in P^\vee, \\ \Delta(e_i) &= e_i \otimes q^{-k_i + k_{i+1}} + 1 \otimes e_i \quad \text{for } i = 1, \dots, n-1, \\ \Delta(f_i) &= f_i \otimes 1 + q^{k_i - k_{i+1}} \otimes f_i \quad \text{for } i = 1, \dots, n-1, \\ \Delta(k_{\bar{1}}) &= k_{\bar{1}} \otimes q^{k_1} + q^{-k_1} \otimes k_{\bar{1}}. \end{aligned}$$

Let U^+ (respectively, U^-) be the subalgebra of $U_q(\mathfrak{q}(n))$ generated by $e_i, e_{\bar{i}}$ ($i = 1, \dots, n-1$) (respectively, $f_i, f_{\bar{i}}$ ($i = 1, \dots, n-1$)), and let U^0 be the subalgebra generated by q^h ($h \in P^\vee$) and $k_{\bar{j}}$ ($j = 1, \dots, n$). In [5], it was shown that the algebra $U_q(\mathfrak{q}(n))$ has the *triangular decomposition*:

$$(1.3) \quad U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U_q(\mathfrak{q}(n)).$$

1.2. The category $\mathcal{O}_{\text{int}}^{\geq 0}$. Hereafter, a $U_q(\mathfrak{q}(n))$ -module is understood as a $U_q(\mathfrak{q}(n))$ -supermodule. A $U_q(\mathfrak{q}(n))$ -module M is called a *weight module* if M has a weight space decomposition $M = \bigoplus_{\mu \in P} M_\mu$, where

$$M_\mu := \{m \in M; q^h m = q^{\mu(h)} m \text{ for all } h \in P^\vee\}.$$

The *set of weights of M* is defined to be

$$\text{wt}(M) = \{\mu \in P; M_\mu \neq 0\}.$$

Definition 1.2. A weight module V is called a highest weight module with highest weight $\lambda \in P$ if V_λ is finite-dimensional and satisfies the following conditions:

- (a) V is generated by V_λ ,
- (b) $e_i v = e_{\bar{i}} v = 0$ for all $v \in V_\lambda$, $i = 1, \dots, n-1$,

As seen in [5], there exists a unique irreducible highest weight module with highest weight $\lambda \in P$ up to parity change, which will be denoted by $V(\lambda)$.

Set

$$\begin{aligned} P^{\geq 0} &= \{\lambda = \lambda_1 \epsilon_1 + \dots + \lambda_n \epsilon_n \in P; \lambda_j \in \mathbf{Z}_{\geq 0} \text{ for all } j = 1, \dots, n\}, \\ \Lambda^+ &= \{\lambda = \lambda_1 \epsilon_1 + \dots + \lambda_n \epsilon_n \in P^{\geq 0}; \lambda_i \geq \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies} \\ &\quad \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \dots, n-1\}. \end{aligned}$$

Note that each element $\lambda \in \Lambda^+$ corresponds to a *strict partition* $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_r > 0)$. Thus we will often call $\lambda \in \Lambda^+$ a strict partition. For the same reason, we call

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in P^{\geq 0}$ a *partition* if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > \lambda_{r+1} = \dots = \lambda_n = 0$. We denote r by $\ell(\lambda)$.

Example 1.3. Let

$$\mathbf{V} = \bigoplus_{j=1}^n \mathbf{F}v_j \oplus \bigoplus_{j=1}^n \mathbf{F}v_{\bar{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. The action of $U_q(\mathfrak{q}(n))$ on \mathbf{V} is given as follows:

$$(1.4) \quad \begin{aligned} e_i v_j &= \delta_{j,i+1} v_i, & e_i v_{\bar{j}} &= \delta_{j,i+1} v_{\bar{i}}, & f_i v_j &= \delta_{j,i} v_{i+1}, & f_i v_{\bar{j}} &= \delta_{j,i} v_{\overline{i+1}}, \\ e_{\bar{i}} v_j &= \delta_{j,i+1} v_{\bar{i}}, & e_{\bar{i}} v_{\bar{j}} &= \delta_{j,i+1} v_i, & f_{\bar{i}} v_j &= \delta_{j,i} v_{\overline{i+1}}, & f_{\bar{i}} v_{\bar{j}} &= \delta_{j,i} v_{i+1}, \\ q^h v_j &= q^{\epsilon_j(h)} v_j, & q^h v_{\bar{j}} &= q^{\epsilon_j(h)} v_{\bar{j}}, & k_{\bar{i}} v_j &= \delta_{j,i} v_{\bar{j}}, & k_{\bar{i}} v_{\bar{j}} &= \delta_{j,i} v_j. \end{aligned}$$

Note that \mathbf{V} is an irreducible highest weight module with highest weight ϵ_1 and $\text{wt}(\mathbf{V}) = \{\epsilon_1, \dots, \epsilon_n\}$.

Definition 1.4. We define the category $\mathcal{O}_{\text{int}}^{\geq 0}$ to be the category of finite-dimensional weight modules M satisfying the following conditions:

- (a) $\text{wt}(M) \subset P^{\geq 0}$,
- (b) for any $\mu \in P^{\geq 0}$ and $i \in \{1, \dots, n\}$ such that $\langle k_i, \mu \rangle = 0$, we have $k_{\bar{i}}|_{M_\mu} = 0$.

Proposition 1.5 ([4, Corollary 1.12]).

- (a) The abelian category $\mathcal{O}_{\text{int}}^{\geq 0}$ is semisimple.
- (b) Any irreducible $U_q(\mathfrak{q}(n))$ -module in $\mathcal{O}_{\text{int}}^{\geq 0}$ appears as a direct summand of tensor products of \mathbf{V} .

1.3. Crystal bases in $\mathcal{O}_{\text{int}}^{\geq 0}$. Let M be a $U_q(\mathfrak{q}(n))$ -module in $\mathcal{O}_{\text{int}}^{\geq 0}$. For $i = 1, 2, \dots, n-1$, and for a weight vector $u \in M_\lambda$, consider the i -string decomposition of u :

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$, $f_i^{(k)} = f_i^k / [k]!$. We define the *even Kashiwara operators* \tilde{e}_i, \tilde{f}_i ($i = 1, \dots, n-1$) by

$$(1.5) \quad \begin{aligned} \tilde{e}_i u &= \sum_{k \geq 1} f_i^{(k-1)} u_k, \\ \tilde{f}_i u &= \sum_{k \geq 0} f_i^{(k+1)} u_k. \end{aligned}$$

On the other hand, we define the *odd Kashiwara operators* $\tilde{k}_{\bar{1}}, \tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}$ by

$$(1.6) \quad \begin{aligned} \tilde{k}_{\bar{1}} &= q^{k_1-1} k_{\bar{1}}, \\ \tilde{e}_{\bar{1}} &= -(e_1 k_{\bar{1}} - q k_{\bar{1}} e_1) q^{k_1-1}, \\ \tilde{f}_{\bar{1}} &= -(k_{\bar{1}} f_1 - q f_1 k_{\bar{1}}) q^{k_2-1}. \end{aligned}$$

Recall that an abstract $\mathfrak{gl}(n)$ -crystal is a set B together with the maps $\tilde{e}_i, \tilde{f}_i: B \rightarrow B \sqcup \{0\}$, $\varphi_i, \varepsilon_i: B \rightarrow \mathbf{Z} \sqcup \{-\infty\}$ ($i = 1, \dots, n-1$), and $\text{wt}: B \rightarrow P$ satisfying the conditions given in [13]. We say that an abstract $\mathfrak{gl}(n)$ -crystal is a $\mathfrak{gl}(n)$ -crystal if it is realized as a crystal basis of a finite-dimensional integrable $U_q(\mathfrak{gl}(n))$ -module. In particular, we have

$$\varepsilon_i(b) = \max\{n \in \mathbf{Z}_{\geq 0}; \tilde{e}_i^n b \neq 0\}, \quad \varphi_i(b) = \max\{n \in \mathbf{Z}_{\geq 0}; \tilde{f}_i^n b \neq 0\}$$

for any b in a $\mathfrak{gl}(n)$ -crystal B .

Definition 1.6. Let $M = \bigoplus_{\mu \in P^{\geq 0}} M_\mu$ be a $U_q(\mathfrak{q}(n))$ -module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. A

crystal basis of M is a triple $(L, B, l_B = (l_b)_{b \in B})$, where

- (a) L is a free \mathbf{A} -submodule of M such that
 - (i) $\mathbf{F} \otimes_{\mathbf{A}} L \xrightarrow{\sim} M$,
 - (ii) $L = \bigoplus_{\mu \in P^{\geq 0}} L_\mu$, where $L_\mu = L \cap M_\mu$,
 - (iii) L is stable under the Kashiwara operators \tilde{e}_i, \tilde{f}_i ($i = 1, \dots, n-1$), $\tilde{k}_{\bar{1}}, \tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}$.
- (b) B is a finite $\mathfrak{gl}(n)$ -crystal together with the maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}: B \rightarrow B \sqcup \{0\}$ such that
 - (i) $\text{wt}(\tilde{e}_{\bar{1}} b) = \text{wt}(b) + \alpha_1$, $\text{wt}(\tilde{f}_{\bar{1}} b) = \text{wt}(b) - \alpha_1$,
 - (ii) for all $b, b' \in B$, $\tilde{f}_{\bar{1}} b = b'$ if and only if $b = \tilde{e}_{\bar{1}} b'$.
- (c) $l_B = (l_b)_{b \in B}$ is a family of non-zero \mathbf{C} -vector spaces such that
 - (i) $l_b \subset (L/qL)_\mu$ for $b \in B_\mu$,
 - (ii) $L/qL = \bigoplus_{b \in B} l_b$,
 - (iii) $\tilde{k}_{\bar{1}} l_b \subset l_b$,
 - (iv) for $i = 1, \dots, n-1, \bar{1}$, we have
 - (1) if $\tilde{e}_i b = 0$ then $\tilde{e}_i l_b = 0$, and otherwise \tilde{e}_i induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{e}_i b}$.
 - (2) if $\tilde{f}_i b = 0$ then $\tilde{f}_i l_b = 0$, and otherwise \tilde{f}_i induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{f}_i b}$.

As proved in [3], for every crystal basis (L, B, l_B) of a $U_q(\mathfrak{q}(n))$ -module M we have $\tilde{e}_{\bar{1}}^2 = \tilde{f}_{\bar{1}}^2 = 0$ as endomorphisms on L/qL .

Example 1.7. Let

$$\mathbf{V} = \bigoplus_{j=1}^n \mathbf{F} v_j \oplus \bigoplus_{j=1}^n \mathbf{F} v_{\bar{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. Set

$$\mathbf{L} = \bigoplus_{j=1}^n \mathbf{A}v_j \oplus \bigoplus_{j=1}^n \mathbf{A}v_{\bar{j}},$$

$l_j = \mathbf{C}v_j \oplus \mathbf{C}v_{\bar{j}}$, and let \mathbf{B} be the crystal graph given below.

$$\boxed{1} \xrightarrow[-\frac{1}{\bar{1}}]{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

Then $(\mathbf{L}, \mathbf{B}, l_{\mathbf{B}} = (l_j)_{j=1}^n)$ is a crystal basis of \mathbf{V} .

The *queer tensor product rule* for the crystal bases of $U_q(\mathfrak{q}(n))$ -modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ is given by the following theorem.

Theorem 1.8. [4, Theorem 2.7]

Let M_j be a $U_q(\mathfrak{q}(n))$ -module in $\mathcal{O}_{\text{int}}^{\geq 0}$ with crystal basis (L_j, B_j, l_{B_j}) ($j = 1, 2$). Set

$$B_1 \otimes B_2 = B_1 \times B_2 \quad \text{and} \quad l_{B_1 \otimes B_2} = (l_{b_1} \otimes l_{b_2})_{b_1 \in B_1, b_2 \in B_2}.$$

Then

$$(L_1 \otimes_{\mathbf{A}} L_2, B_1 \otimes B_2, l_{B_1 \otimes B_2})$$

is a crystal basis of $M_1 \otimes_{\mathbf{F}} M_2$, where the action of the Kashiwara operators on $B_1 \otimes B_2$ are given as follows.

$$(1.7) \quad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned}$$

$$(1.8) \quad \begin{aligned} \tilde{e}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \text{wt } b_2 \rangle = \langle k_2, \text{wt } b_2 \rangle = 0, \\ b_1 \otimes \tilde{e}_{\bar{1}} b_2 & \text{otherwise,} \end{cases} \\ \tilde{f}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \text{wt } b_2 \rangle = \langle k_2, \text{wt } b_2 \rangle = 0, \\ b_1 \otimes \tilde{f}_{\bar{1}} b_2 & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 1.9. An abstract $\mathfrak{q}(n)$ -crystal is a $\mathfrak{gl}(n)$ -crystal together with the maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}: B \rightarrow B \sqcup \{0\}$ satisfying the following conditions:

- (a) $\text{wt}(B) \subset P^{\geq 0}$,
- (b) $\text{wt}(\tilde{e}_{\bar{1}}b) = \text{wt}(b) + \alpha_1$, $\text{wt}(\tilde{f}_{\bar{1}}b) = \text{wt}(b) - \alpha_1$,
- (c) for all $b, b' \in B$, $\tilde{f}_{\bar{1}}b = b'$ if and only if $b = \tilde{e}_{\bar{1}}b'$.
- (d) if $3 \leq i \leq n-1$, we have

- (i) the operators $\tilde{e}_{\overline{1}}$ and $\tilde{f}_{\overline{1}}$ commute \tilde{e}_i and \tilde{f}_i .
- (ii) if $\tilde{e}_{\overline{1}}b \in B$, then $\varepsilon_i(\tilde{e}_{\overline{1}}b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_{\overline{1}}b) = \varphi_i(b)$.

For an abstract $\mathfrak{q}(n)$ -crystal B and an element $b \in B$, we denote by $C(b)$ the connected component of b in B .

Let B_1 and B_2 be abstract $\mathfrak{q}(n)$ -crystals. The *tensor product* $B_1 \otimes B_2$ of B_1 and B_2 is defined to be the $\mathfrak{gl}(n)$ -crystal $B_1 \otimes B_2$ together with the maps $\tilde{e}_{\overline{1}}, \tilde{f}_{\overline{1}}$ defined by (1.8). Then it is an abstract $\mathfrak{q}(n)$ -crystal. Note that \otimes satisfies the associativity axiom on the set of abstract $\mathfrak{q}(n)$ -crystals.

The next lemma follows directly from (1.7) and (1.8). It will be used in Section 2 and Section 4.

Lemma 1.10. *Let B_j ($j = 1, \dots, N$) be abstract $\mathfrak{q}(n)$ -crystals, and let $b_j \in B_j$ ($j = 1, \dots, N$).*

- (a) *For $i \in \{1, \dots, n-1, \overline{1}\}$, suppose that $\tilde{f}_i(b_1 \otimes \dots \otimes b_N) = b_1 \otimes \dots \otimes b_{k-1} \otimes \tilde{f}_i b_k \otimes b_{k+1} \otimes \dots \otimes b_N$ for some $1 \leq k \leq N$. Then for any positive integers j and m such that $1 \leq j \leq k \leq m$, we have*

$$\tilde{f}_i(b_j \otimes \dots \otimes b_m) = b_j \otimes \dots \otimes b_{k-1} \otimes \tilde{f}_i b_k \otimes b_{k+1} \otimes \dots \otimes b_m.$$

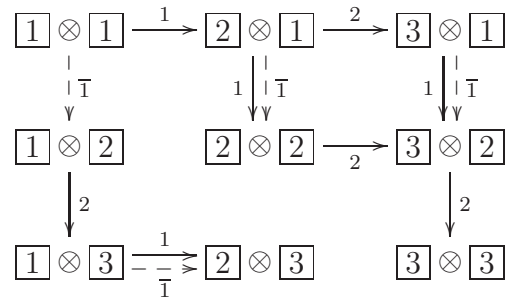
- (b) *For $i \in \{1, \dots, n-1, \overline{1}\}$, suppose that $\tilde{e}_i(b_1 \otimes \dots \otimes b_N) = b_1 \otimes \dots \otimes b_{k-1} \otimes \tilde{e}_i b_k \otimes b_{k+1} \otimes \dots \otimes b_N$ for some $1 \leq k \leq N$. Then for any positive integers j and m such that $1 \leq j \leq k \leq m$, we have*

$$\tilde{e}_i(b_j \otimes \dots \otimes b_m) = b_j \otimes \dots \otimes b_{k-1} \otimes \tilde{e}_i b_k \otimes b_{k+1} \otimes \dots \otimes b_m.$$

Example 1.11. (a) If (L, B, l_B) is a crystal basis of a $U_q(\mathfrak{q}(n))$ -module M in the category $\mathcal{O}_{\text{int}}^{\geq 0}$, then B is an abstract $\mathfrak{q}(n)$ -crystal.

(b) The crystal graph \mathbf{B} is an abstract $\mathfrak{q}(n)$ -crystal.

(c) By the tensor product rule, $\mathbf{B}^{\otimes N}$ is an abstract $\mathfrak{q}(n)$ -crystal. When $n = 3$, the $\mathfrak{q}(n)$ -crystal structure of $\mathbf{B} \otimes \mathbf{B}$ is given below.



Let W be the Weyl group of $\mathfrak{gl}(n)$ and let B be an abstract $\mathfrak{q}(n)$ -crystal. For $i = 1, \dots, n-1$, we define the automorphism S_i on B by

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, \text{wt } b \rangle} b & \text{if } \langle h_i, \text{wt } b \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \text{wt } b \rangle} b & \text{if } \langle h_i, \text{wt } b \rangle \leq 0. \end{cases}$$

As shown in [12], there exists a unique (well-defined) action $S : W \rightarrow \text{Aut } B$ such that $S_{s_i} = S_i$. Here s_i is the simple reflection given by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ ($\lambda \in \mathfrak{h}^*$). Note that $\text{wt}(S_w b) = w(\text{wt}(b))$ for any $w \in W$ and $b \in B$.

For $i = 1, \dots, n-1$, set

$$(1.9) \quad w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}.$$

Then w_i is the shortest element in W such that $w_i(\alpha_i) = \alpha_1$. We define the *odd Kashiwara operators* \tilde{e}_i, \tilde{f}_i ($i = 2, \dots, n-1$) by

$$\tilde{e}_i = S_{w_i^{-1}} \tilde{e}_1 S_{w_i}, \quad \tilde{f}_i = S_{w_i^{-1}} \tilde{f}_1 S_{w_i}.$$

Definition 1.12. Let B be an abstract $\mathfrak{q}(n)$ -crystal and $1 \leq a \leq n$.

- (a) An element $b \in B$ is called a $\mathfrak{gl}(a)$ -highest weight vector if $\tilde{e}_i b = 0$ for $1 \leq i < a$.
- (b) An element $b \in B$ is called a $\mathfrak{q}(a)$ -highest weight vector if $\tilde{e}_i b = \tilde{e}_i^- b = 0$ for $1 \leq i < a$.
- (c) An element $b \in B$ is called a $\mathfrak{q}(n)$ -lowest weight vector if $S_{w_0} b$ is a $\mathfrak{q}(n)$ -highest weight vector, where w_0 is the longest element of W .

The $\mathfrak{q}(n)$ -highest (respectively, lowest) weight vectors will be called *highest* (respectively, *lowest*) *weight vectors*. We denote by $\mathcal{HW}(\lambda)$ (respectively, $\mathcal{LW}(\lambda)$) the set of highest (respectively, lowest) weight vectors of weight λ in $\mathbf{B}^{\otimes |\lambda|}$. The description of $\mathcal{HW}(\lambda)$ (and hence of $\mathcal{LW}(\lambda)$) is given by the following proposition (see Theorem 4.6 (c) in [4])

Proposition 1.13. An element b_0 in $\mathbf{B}^{\otimes N}$ is a highest weight vector if and only if $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ for some j and some highest weight vector b in $\mathbf{B}^{\otimes (N-1)}$ such that $\text{wt}(b_0) = \text{wt}(b) + \epsilon_j$ is a strict partition.

The following theorem is part of the main result in [4].

Theorem 1.14. (a) For any $\lambda \in \Lambda^+$, there exists a crystal basis (L, B, l_B) of the irreducible highest weight module $V(\lambda)$ such that

- (i) $B_\lambda = \{b_\lambda\}$,
- (ii) B is connected.

Moreover, such a crystal basis is unique. In particular B depends only on λ as an abstract $\mathfrak{q}(n)$ -crystal. Hence we may write $B = B(\lambda)$.

- (b) The $\mathfrak{q}(n)$ -crystal $B(\lambda)$ has a unique highest weight vector b_λ and unique lowest weight vector l_λ .

We close this section with preparatory statements that will be useful in the following sections.

Lemma 1.15. *Let $a \in \mathbf{B}$ and $b \in \mathbf{B}^{\otimes N}$. Then $a \otimes b$ is a lowest weight vector if and only if b is a lowest weight vector and $\epsilon_a + \text{wt } b \in w_0\Lambda^+$.*

Proof. Let $a \otimes b \in \mathbf{B} \otimes \mathbf{B}^{\otimes N}$ be a $\mathfrak{gl}(n)$ -lowest weight vector. Then b is a $\mathfrak{gl}(n)$ -lowest weight vector in $\mathbf{B}^{\otimes N}$ and hence $S_{w_0}b$ is the unique $\mathfrak{gl}(n)$ -highest weight vector in the $\mathfrak{gl}(n)$ -connected component containing b in $\mathbf{B}^{\otimes N}$. By (a) of Lemma 3.3 in [4], it follows that $S_{w_0}(a \otimes b) = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} S_{w_0}b$ for some $1 \leq j \leq n$. Comparing the weights, we have $j = n - a + 1$.

Now let $a \otimes b$ be a $\mathfrak{q}(n)$ -lowest weight vector. Then $a \otimes b$ is a $\mathfrak{gl}(n)$ -lowest weight vector and hence $S_{w_0}(a \otimes b) = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{n-a} S_{w_0}b$. By Proposition 1.13, $S_{w_0}b$ is a $\mathfrak{q}(n)$ -highest weight vector and $w_0(\epsilon_a + \text{wt } b) \in \Lambda^+$.

Conversely, let $S_{w_0}b$ is a $\mathfrak{q}(n)$ -highest weight vector and $w_0(\epsilon_a + \text{wt } b) \in \Lambda^+$. It is straightforward to check that $a \otimes b$ is a $\mathfrak{gl}(n)$ -lowest weight vector. Hence $S_{w_0}(a \otimes b) = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{n-a} S_{w_0}b$. Again, by Proposition 1.13, we conclude that $S_{w_0}(a \otimes b)$ is a $\mathfrak{q}(n)$ -highest weight vector. \square

The following corollary immediately follows from the preceding lemma.

Corollary 1.16. *Let b_1, \dots, b_N be elements in \mathbf{B} . Then $b_1 \otimes \cdots \otimes b_N$ is a lowest weight vector in $\mathbf{B}^{\otimes N}$ if and only if $\text{wt}(b_k) + \cdots + \text{wt}(b_N) \in w_0\Lambda^+$ for all $k = 1, \dots, N$.*

Definition 1.17. *A finite sequence of positive integers $x = x_1 \cdots x_N$ is called a strict reverse lattice permutation if for $1 \leq k \leq N$ and $2 \leq i \leq n$, the number of occurrences of i is strictly greater than the number of occurrences of $i - 1$ in $x_k \cdots x_N$ as soon as $i - 1$ appears in $x_k \cdots x_N$.*

Then we can rephrase Corollary 1.16 as follows.

Corollary 1.18. *A vector $b_1 \otimes \cdots \otimes b_N \in \mathbf{B}^{\otimes N}$ is a lowest weight vector if and only if it is a strict reverse lattice permutation.*

2. SEMISTANDARD DECOMPOSITION TABLEAUX

2.1. Semistandard decomposition tableaux. For a strict partition $\lambda = (\lambda_1, \dots, \lambda_n)$, we set $|\lambda| := \lambda_1 + \cdots + \lambda_n$. Recall that $\ell(\lambda)$ is the number of nonzero λ_i 's.

Definition 2.1.

- (a) *The shifted Young diagram of shape λ is an array of square cells in which the i -th row has λ_i cells, and is shifted $i - 1$ units to the right with respect to the top row. In this case, we say that λ is a shifted shape.*

(b) A word $u = u_1 \cdots u_N$ is a hook word if there exists $1 \leq k \leq N$ such that

$$(2.1) \quad u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_N.$$

Every hook word has the decreasing part $u \downarrow = u_1 \cdots u_k$, and the increasing part $u \uparrow = u_{k+1} \cdots u_N$ (note that the decreasing part is always nonempty).

(c) A semistandard decomposition tableau of a shifted shape $\lambda = (\lambda_1, \dots, \lambda_n)$ is a filling T of λ with elements of $\{1, 2, \dots, n\}$ such that:

(i) the word v_i formed by reading the i -th row from left to right is a hook word of length λ_i ,

(ii) v_i is a hook subword of maximal length in $v_{i+1}v_i$ for $1 \leq i \leq \ell(\lambda) - 1$.

(d) The reading word of a semistandard decomposition tableau T is

$$\text{read}(T) = v_{\ell(\lambda)} v_{\ell(\lambda)-1} \cdots v_1.$$

Remark 2.2.

- (i) Our definition of a hook word, and hence of a semistandard decomposition tableau, is different from the one used in [20], where $u \downarrow$ is assumed to be strictly decreasing, while $u \uparrow$ is weakly increasing. Later, we will consider the $\mathfrak{q}(n)$ -crystal structure on the set of all semistandard decomposition tableaux of a shifted shape λ . Then the highest weight vectors and the lowest weight vectors have simpler forms in our choice than the ones in [20] (see Example 2.4 and Remark 2.6).
- (ii) The term “hook word” in [21] refers to a word u with strictly decreasing $u \downarrow$ and strictly increasing $u \uparrow$. This definition leads to the notion of *standard decomposition tableaux*.
- (iii) If there is any, the way to view a word as a semistandard decomposition tableau is unique.

We have an alternative criterion to determine whether a filling of shifted shape λ (equivalently, its reading word) is a semistandard decomposition tableau or not.

Proposition 2.3. *Let $u = u_1 \cdots u_\ell$ and $u' = u'_1 \cdots u'_{\ell'}$ be hook words with $1 \leq \ell' < \ell$. Then $u'u$ is a semistandard decomposition tableau if and only if for $1 \leq i \leq j \leq \ell'$,*

- (a) if $u_i \leq u'_j$, then $i \neq 1$ and $u'_{i-1} < u'_j$,
- (b) if $u_i > u'_j$, then $u_i \geq u_{j+1}$.

This is equivalent to saying that none of the following conditions holds.

- (i) $u_1 \leq u'_i$ ($1 \leq i \leq \ell'$),
- (ii) $u'_i \geq u'_j \geq u_{i+1}$ ($i < j \leq \ell'$),
- (iii) $u'_j < u_i < u_{j+1}$ ($i \leq j \leq \ell'$).

Proof. Assume that $u'u$ is a semistandard decomposition tableau.

If $u_1 \leq u'_i$ for $1 \leq i \leq \ell'$, then $u'_i u_1 u_2 \cdots u_\ell$ is a hook subword of $u'u$ with length $\ell + 1$, which is a contradiction.

If $u'_i \geq u'_j \geq u_{i+1}$ for $i < j \leq \ell'$, then $u'_i \in u' \downarrow$ and hence $u'_1 \cdots u'_i u'_j u_{i+1} \cdots u_\ell$ is a hook subword of $u'u$ of length $\ell + 1$, which is a contradiction.

If $u'_j < u_i < u_{j+1}$ for $i \leq j \leq \ell'$, then $u_{j+1} \in u \uparrow$ and hence $u'_1 \cdots u'_j u_i u_{j+1} \cdots u_\ell$ is a hook subword of $u'u$ of length $\ell + 1$, which is a contradiction.

Now assume that none of (i), (ii), (iii) holds. Suppose that v is a hook subword of $u'u$ of length greater than ℓ . Let x be the first letter in $v \cap u$ and let x' be the last letter in $v \cap u'$.

Case 1: $x' \geq x$.

Note that x cannot be u_1 , since (i) does not hold. Let x be u_{i+1} for some $i \in \{1, 2, \dots, \ell - 1\}$ and let x' be u'_j for some $j \in \{1, 2, \dots, \ell'\}$. Then the length of $v \cap u$ is less than or equal to $\ell - i$ and hence the length of $v \cap u'$ is greater than or equal to $i + 1$, which implies $i < j$. Moreover, since $i + 1 \geq 2$, $v \cap u'$ contains another letter besides u'_j . Let u'_k be the second last letter in $v \cap u'$ ($k < j$). Then we have $k \geq i$, since the length of $v \cap u'$ is greater than or equal to $i + 1$. Because $u'_j \geq u_{i+1}$, we have $u'_j \in v \downarrow$ and hence $u'_k \in v \downarrow$. Thus we get $u'_k \geq u'_j$. It follows that $u'_k \in u' \downarrow$ and hence $u'_i \geq u'_k \geq u'_j \geq u_{i+1}$, which is a contradiction to (ii).

Case 2: $x' < x$.

Let x be u_i for some $i \in \{1, 2, \dots, \ell\}$ and let x' be u'_j for some $j \in \{1, 2, \dots, \ell'\}$. Note that the length of $v \cap u'$ is less than or equal to j and hence the length of $v \cap u$ is greater than or equal to $\ell - j + 1$, because the length of v is greater than ℓ . Moreover we have $i \leq j$, and $v \cap u$ contains another letter in u besides u_i . Let u_k be the second letter in $v \cap u$ ($k > i$). Note that $k \leq j + 1$ since the length of $v \cap u$ is greater than or equal to $\ell - j + 1$. On the other hand, $u'_j < u_i$ implies $u_i, u_k \in v \uparrow$. Thus we have $u_i < u_k$. It follows that $u_k \in u \uparrow$ so that $u'_j < u_i < u_k \leq u_{j+1}$, which is a contradiction to (iii). \square

If T is a semistandard decomposition tableau of shifted shape λ , we write $\text{sh}(T) = \lambda$. Let $\mathbf{B}(\lambda)$ denote the set of all semistandard decomposition tableau T with $\text{sh}(T) = \lambda$. For every $\lambda \in \Lambda^+$, we have the following embedding

$$\text{read} : \mathbf{B}(\lambda) \rightarrow \mathbf{B}^{\otimes |\lambda|}, \quad T \mapsto \text{read}(T).$$

Using this embedding, we identify $\mathbf{B}(\lambda)$ with a subset in $\mathbf{B}^{\otimes |\lambda|}$ and define the action of the Kashiwara operators $\tilde{e}_i, \tilde{e}_{\bar{i}}, \tilde{f}_i, \tilde{f}_{\bar{i}}$ on the elements in $\mathbf{B}(\lambda)$. The question is whether the set $\mathbf{B}(\lambda)$ is closed under these operators.

For a strict partition λ with $\ell(\lambda) = r$, set

$$\begin{aligned} T^\lambda &:= (1^{\lambda_r})(2^{\lambda_r} 1^{\lambda_{r-1}-\lambda_r}) \cdots ((r-k+1)^{\lambda_r} (r-k)^{\lambda_{r-1}-\lambda_r} \cdots 1^{\lambda_k-\lambda_{k+1}}) \\ &\quad \cdots (r^{\lambda_r} (r-1)^{\lambda_{r-1}-\lambda_r} \cdots 1^{\lambda_1-\lambda_2}), \\ L^\lambda &:= (n-r+1)^{\lambda_r} \cdots (n-k+1)^{\lambda_k} \cdots n^{\lambda_1}. \end{aligned}$$

Then we have $S_{w_0}T^\lambda = L^\lambda$.

Example 2.4. Let $n = 4$ and $\lambda = (6, 4, 2, 1)$. Then we have

$$T^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 2 & 2 & 1 & 1 \\ \hline & 3 & 2 & 1 & 1 & \\ \hline & & 2 & 1 & & \\ \hline & & & 1 & & \\ \hline \end{array} \quad \text{and} \quad L^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 4 & 4 & 4 & 4 & 4 \\ \hline & 3 & 3 & 3 & 3 & \\ \hline & & 2 & 2 & & \\ \hline & & & 1 & & \\ \hline \end{array}.$$

Our first main result is given in the following theorem.

Theorem 2.5. *Let λ be a strict partition with $\ell(\lambda) = r$.*

- (a) *The set $\mathbf{B}(\lambda) \cup \{0\}$ is closed under the action of the Kashiwara operators. In particular, $\mathbf{B}(\lambda)$ becomes an abstract $\mathfrak{q}(n)$ -crystal.*
- (b) *The element T^λ is a unique highest weight vector in $\mathbf{B}(\lambda)$ and L^λ is a unique lowest weight vector in $\mathbf{B}(\lambda)$.*
- (c) *The abstract $\mathfrak{q}(n)$ -crystal $\mathbf{B}(\lambda)$ is isomorphic to $B(\lambda)$, the crystal of the irreducible highest weight module $V(\lambda)$.*

Proof. (a) *Step 1:* Let $u = u_1 \cdots u_N$ be a hook word such that

$$u_1 \geq u_2 \geq \cdots \geq u_k < u_{k+1} < \cdots < u_N.$$

We will prove that $\tilde{f}_i u, \tilde{e}_i u$ ($i = 1, \dots, n-1, \bar{1}$) are hook words, when they are nonzero.

Assume that $\tilde{f}_i u \neq 0$ for some $i \in \{1, \dots, n-1\}$. Note that u can be regarded as a semistandard tableau of shape $k\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{N-k+1}$. Since the set of semistandard tableaux of a skew shape is closed under the action of the even Kashiwara operators, $\tilde{f}_i u$ is a semistandard tableau of shape $k\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{N-k+1}$ and hence it is a hook word. For the same reason, we deduce that $\tilde{e}_i u$ is a hook word for $i = 1, 2, \dots, n-1$, unless it is zero.

Assume that $\tilde{f}_{\bar{1}} u \neq 0$. Then we have $u_k = 1, u_j > 2$ for $j \geq k+1$ and hence

$$\tilde{f}_{\bar{1}} u = u_1 \cdots u_{k-1} 2u_{k+1} \cdots u_N.$$

Note that if $u_{k-1} = 1$, then $u_1 \geq \cdots \geq u_{k-1} < 2 < u_{k+1} < \cdots < u_N$, and if $u_{k-1} \geq 2$ then $u_1 \geq \cdots \geq u_{k-1} \geq 2 < u_{k+1} < \cdots < u_N$. In both cases, $\tilde{f}_{\bar{1}} u$ is a hook word.

Assume that $\tilde{e}_{\bar{1}} u \neq 0$. Since $u_k = \min\{u_j; j = 1, \dots, N\}$, we have $u_k \leq 2$. If $u_k = 2$, then $u_j > 2$ for $j \geq k+1$ and hence $\tilde{e}_{\bar{1}} u = u_1 \cdots u_{k-1} 1u_{k+1} \cdots u_N$. It follows that $\tilde{e}_{\bar{1}} u$ is a hook word. If $u_k = 1$, then $u_{k+1} = 2$, because $\tilde{e}_{\bar{1}} u \neq 0$. Hence we get $\tilde{e}_{\bar{1}} u = u_1 \cdots u_{k-1} 11u_{k+2} \cdots u_n$, which is a hook word.

Let v_j be the reading word of the j -th row of a semistandard decomposition tableau u . By Lemma 1.10, we know that $\tilde{f}_i u = v_r \cdots \tilde{f}_i v_a \cdots v_1$ for some $1 \leq a \leq r$, and $\tilde{e}_i u = v_r \cdots \tilde{e}_i v_b \cdots v_1$ for some $1 \leq b \leq r$. Hence we conclude all the rows of $\tilde{f}_i u$ and $\tilde{e}_i u$ are again hook words.

Step 2: Let $u = u_1 \cdots u_N$ be a semistandard decomposition tableau of shifted shape λ . We will show that $\tilde{f}_i u, \tilde{e}_i u$ ($i = 1, \dots, n-1, \bar{1}$) satisfy the condition in Definition 2.1 (c) (ii), when they are nonzero. We will prove our claim in four separate cases.

Case 1: For an $i \in \{1, \dots, n-1\}$, assume that $\tilde{f}_i u = u_1 \cdots u_{t-1} u'_t u_{t+1} \cdots u_N$, where $u_t = i$ and $u'_t = \tilde{f}_i u_t = i+1$. Let $v' = u_{j_1} \cdots u_{j_{\ell-1}} u'_t u_{j_{\ell+1}} \cdots u_{j_r}$ be a hook subword of $\tilde{f}_i u$. By Lemma 1.10, taking two consecutive rows of u which contains u_t , one can assume that $\lambda_3 = 0$ from the beginning. Then it is enough to show that there exists a hook subword v of u of length r .

(i) Suppose $u'_t \in v' \downarrow$. Let $u'_t = u_{j_{\ell+1}} = u_{j_{\ell+2}} = \cdots = u_{j_{\ell+s}} = i+1$ and $u_{j_{\ell+s+1}} \neq i+1$ for some $s \geq 0$. Here, we regard $u_{j_{\ell+s+1}}$ as the empty word, if $\ell+s = r$.

If $s = 0$, then replacing u'_t by u_t in v' , we get a subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u of length r . Since we have

$$\begin{cases} u_{j_{\ell-1}} > u_t \geq u_{j_{\ell+1}} & \text{if } u_{j_{\ell+1}} < u'_t, \\ u_{j_{\ell-1}} > u_t < u_{j_{\ell+1}} < \cdots < u_{j_r} & \text{if } u_{j_{\ell+1}} > u'_t, \end{cases}$$

v is a hook subword of u of length r .

Assume that $s \geq 1$. Since \tilde{f}_i acts on u_t , we know that for each $p = 1, 2, \dots, s$, there exists v_p between u_t and $u_{j_{\ell+p}}$ in u such that $v_p = i$. We can assume that $v_p \neq v_{p'}$ for $p \neq p'$. Replacing $u_{j_{\ell+p}}$ by v_p for each $1 \leq p \leq s$ and u'_t by u_t in v' , we obtain a subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t v_1 \cdots v_s u_{j_{\ell+s+1}} \cdots u_{j_r}$ of u .

If $\ell+s = r$, then we have $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t v_1 \cdots v_s$ and it is a hook word.

Assume that $\ell+s < r$. If $u_{j_{\ell+s+1}} \in v' \downarrow$, then $u_{j_{\ell+s+1}} \leq i$, and hence

$$u_{j_{\ell-1}} > u_t = v_1 = \cdots = v_s \geq u_{j_{\ell+s+1}}.$$

If $u_{j_{\ell+s+1}} \in v' \uparrow$, then $u_{j_{\ell+s+1}} > i+1$ and hence

$$u_{j_{\ell-1}} > u_t = v_1 = \cdots = v_s < u_{j_{\ell+s+1}} < \cdots < u_{j_r}.$$

In both cases, v is a hook subword of u of length r .

(ii) Suppose $u'_t \in v' \uparrow$. Then $j_{\ell-1} \geq 1$ and $u_{j_{\ell-1}} \leq i$. If $u_{j_{\ell-1}} < i$, replacing u'_t by u_t , we obtain a hook subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u of length r .

If $u_{j_{\ell-1}} = i$, then we know that there exists u_q between $u_{j_{\ell-1}}$ and u_t in u such that $u_q = i+1$. Replace u'_t by u_q in v' . Then we have a subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_q u_{j_{\ell+1}} \cdots u_{j_r}$ of u such that

$$u_{j_{\ell-1}} = i < u_q = i+1 = u'_t < u_{j_{\ell+1}}.$$

Thus v is a hook subword of u of length r .

Case 2: For an $i \in \{1, \dots, n-1\}$, assume that $\tilde{e}_i u = u_1 \cdots u_{t-1} u'_t u_{t+1} \cdots u_N$, where $u_t = i+1$ and $u'_t = \tilde{e}_i u_t = i$. Let $v' = u_{j_1} \cdots u_{j_{\ell-1}} u'_t u_{j_{\ell+1}} \cdots u_{j_r}$ be a hook subword of $\tilde{e}_i u$. We will show that there exists a hook subword v of u of length r .

(i) Suppose $u'_t \in v' \downarrow$. Then we have $u_{j_p} \geq i$ for $p = 1, \dots, j_{\ell-1}$. Let $u'_t = u_{j_{\ell-1}} = u_{j_{\ell-2}} = \dots = u_{j_{\ell-s}} = i$ and $u_{j_{\ell-s-1}} > i$ for some $s \geq 0$. Here we regard $u_{j_{\ell-s-1}}$ as the empty word, if $\ell - s = 1$.

If $s = 0$, then replacing u'_t by u_t , we obtain a hook subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u of length r , since

$$\begin{cases} u_{j_{\ell-1}} \geq u_t > u_{j_{\ell+1}} & \text{if } u_{j_{\ell+1}} \leq i, \\ u_{j_{\ell-1}} \geq u_t = u_{j_{\ell+1}} & \text{if } u_{j_{\ell+1}} = i + 1, \\ u_{j_{\ell-1}} \geq u_t < u_{j_{\ell+1}} < \dots < u_{j_r} & \text{if } u_{j_{\ell+1}} > i + 1. \end{cases}$$

Assume $s \geq 1$. Since \tilde{e}_i acts on u_t , we know that for each $p = 1, 2, \dots, s$, there exists v_p between $u_{j_{\ell-p}}$ and u_t in u such that $v_p = i + 1$. We can assume that $v_p \neq v_{p'}$ for $p \neq p'$. Replace $u_{j_{\ell-p}}$ by v_p for each $0 \leq p \leq s$ and u'_t by u_t in v' . Then we get a subword $v = u_{j_1} \cdots u_{j_{\ell-s-1}} v_s \cdots v_1 u_t \cdots u_{j_{\ell+1}} \cdots u_{j_r}$ of u such that

$$u_{j_1} \geq \dots \geq u_{j_{\ell-s-1}} \geq v_s = \dots = v_1 = u_t.$$

Since

$$\begin{cases} u_t > u_{j_{\ell+1}} & \text{if } u_{j_{\ell+1}} \leq i, \\ u_t = u_{j_{\ell+1}} & \text{if } u_{j_{\ell+1}} = i + 1, \\ u_t < u_{j_{\ell+1}} < \dots < u_{j_r} & \text{if } u_{j_{\ell+1}} > i + 1, \end{cases}$$

v is a hook subword of u of length r .

(ii) Suppose $u'_t \in v' \uparrow$. If $u_{j_{\ell+1}} = i + 1$, then we know that there exists u_q between u_t and $u_{j_{\ell+1}}$ in u such that $u_q = i$. Replace u'_t by u_q in v' . Then we have a hook subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_q u_{j_{\ell+1}} \cdots u_{j_r}$ of u .

If $u_{j_{\ell+1}} > i + 1$, then replacing u'_t by u_t in v' , we have a word $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ such that

$$u_{j_{\ell-1}} < u_t < u_{j_{\ell+1}} < \dots < u_{j_r}.$$

Hence v is a hook subword of u of length r .

Case 3: Let $\tilde{f}_{\overline{1}} u = u_1 \cdots u_{t-1} u'_t u_{t+1} \cdots u_N$, where $u_t = 1$ and $u'_t = \tilde{f}_{\overline{1}} u_t = 2$. We have $u_j \geq 3$ for $j \geq t + 1$. Let $v' = u_{j_1} \cdots u_{j_{\ell-1}} u'_t u_{j_{\ell+1}} \cdots u_{j_r}$ be a hook subword of $\tilde{f}_{\overline{1}} u$ of length r .

If $u'_t \in v' \downarrow$, then we have $u_{j_{\ell+1}} \in v' \uparrow$. Replacing u'_t by u_t , we obtain a subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u such that

$$u_{j_1} \geq \dots \geq u_{j_{\ell-1}} > u_t < u_{j_{\ell+1}} < \dots < u_{j_r}$$

of length r . It follows that v is a hook subword of u .

If $u'_t \in v' \uparrow$, then $1 = u_{j_{\ell-1}} \in v' \downarrow$. Replace u'_t by u_t in v' . Then we obtain a hook subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u of length r , since

$$u_{j_1} \geq \dots \geq u_{j_{\ell-1}} = u_t < u_{j_{\ell+1}} < \dots < u_{j_r}.$$

Case 4: Let $\tilde{e}_{\overline{1}}u = u_1 \cdots u_{t-1} u'_t u_{t+1} \cdots u_N$, where $u_t = 2$ and $u'_t = \tilde{e}_{\overline{1}}u_t = 1$. We have $u_j \geq 3$ for $j \geq t+1$. Let $v' = u_{j_1} \cdots u_{j_{\ell-1}} u'_t u_{j_{\ell+1}} \cdots u_{j_r}$ be a hook subword of $\tilde{e}_{\overline{1}}u$ of length r . Note that $u'_t \in u' \downarrow$. Replace u'_t by u_t in v' , we obtain a hook subword $v = u_{j_1} \cdots u_{j_{\ell-1}} u_t u_{j_{\ell+1}} \cdots u_{j_r}$ of u such that

$$\begin{cases} u_{j_1} \geq \cdots \geq u_{j_{\ell-1}} < u_t < u_{j_{\ell+1}} < \cdots < u_{j_r} & \text{if } u_{j_{\ell-1}} = 1, \\ u_{j_1} \geq \cdots \geq u_{j_{\ell-1}} \geq u_t < u_{j_{\ell+1}} < \cdots < u_{j_r} & \text{if } u_{j_{\ell-1}} \geq 2 \end{cases}$$

of length r , as desired.

(b) It is straightforward to verify that L^λ is a semistandard decomposition tableau of shape λ and $\text{wt}(L^\lambda) = w_0\lambda$. Let $\lambda' = \lambda - \epsilon_r$. By induction on $|\lambda|$, we know that

$$L^{\lambda'} = (n-r+1)^{\lambda_r-1} (n-r+2)^{\lambda_{r-1}} \cdots (n-k+1)^{\lambda_k} \cdots n^{\lambda_1}$$

is a unique lowest weight vector in $\mathbf{B}(\lambda')$. Note that if $u_1 u_2 \cdots u_N \in \mathbf{B}(\lambda)$, then $u_2 \cdots u_N \in \mathbf{B}(\lambda')$. Thus we have a crystal embedding

$$\mathbf{B}(\lambda) \hookrightarrow \mathbf{B} \otimes \mathbf{B}(\lambda')$$

given by $u_1 u_2 \cdots u_N \mapsto u_1 \otimes u_2 \cdots u_N$.

Let $j \otimes L^{\lambda'}$ be a lowest weight vector in $\mathbf{B} \otimes \mathbf{B}(\lambda')$. Then, by Lemma 1.15, we get $\lambda'_{n-j} > \lambda'_{n-j+1} + 1$ and hence $\lambda'_{n-j} \geq 2$. There are three possibilities: (i) $j = n-r$ and $\lambda'_r \geq 2$, (ii) $j > n-r+1$, and (iii) $j = n-r+1$.

If $j = n-r$ and $\lambda'_r \geq 2$, then the word $(n-r)(n-r+1)^{\lambda'_r}$ formed by the first $(\lambda'_r + 1)$ -many letters of $j \otimes L^{\lambda'}$ is not a hook word. Hence $j \otimes L^{\lambda'} \notin \mathbf{B}(\lambda)$.

If $j > n-r+1$, then $j \otimes (n-r+2)^{\lambda_{r-1}}$ is a hook subword of the word formed by the first $(\lambda_r + \lambda_{r-1})$ -many letters of $j \otimes L^{\lambda'}$ and its length is $\lambda_{r-1} + 1$. It follows that $j \otimes L^{\lambda'} \notin \mathbf{B}(\lambda)$ for $j > n-r+1$.

We conclude that $(n-r+1) \otimes L^{\lambda'} = L^\lambda$ is the only lowest weight vector in $\mathbf{B}(\lambda)$. Hence $T^\lambda = S_{w_0} L^\lambda$ is the only highest weight vector in $\mathbf{B}(\lambda)$.

(c) By (a) and (b), the $\mathfrak{q}(n)$ -crystal $\mathbf{B}(\lambda)$ is connected, which proves our assertion. \square

Remark 2.6. One can show that the set of semistandard decomposition tableaux of a shifted shape given in [20] (which is different from the one in this paper) also admits a $\mathfrak{q}(n)$ -crystal structure by reading a semistandard decomposition tableau from right to left, from top to bottom. In this setting, the highest weight vector and lowest weight vectors in the set of semistandard decomposition tableaux of shifted shape

$\lambda = (6, 4, 2, 1)$ are given by

$$\overline{T}^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 2 & 1 & 1 & 1 \\ \hline & 3 & 2 & 1 & 2 & \\ \hline & & 2 & 1 & & \\ \hline & & & 1 & & \\ \hline \end{array} \quad \text{and} \quad \overline{L}^\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 3 & 2 & 1 & 3 & 4 \\ \hline & 4 & 3 & 2 & 4 & \\ \hline & & 4 & 3 & & \\ \hline & & & 4 & & \\ \hline \end{array}.$$

Example 2.7. (a) Since any word of length 2 is a hook word, we obtain $\mathbf{B} \otimes \mathbf{B} \simeq \mathbf{B}(2\epsilon_1)$. Thus the crystal in Example 1.11 (c) is the $\mathfrak{q}(3)$ -crystal $\mathbf{B}(2\epsilon_1)$.
 (b) In Figure 1, we illustrate the $\mathfrak{q}(3)$ -crystal $\mathbf{B}(3\epsilon_1 + \epsilon_2)$.

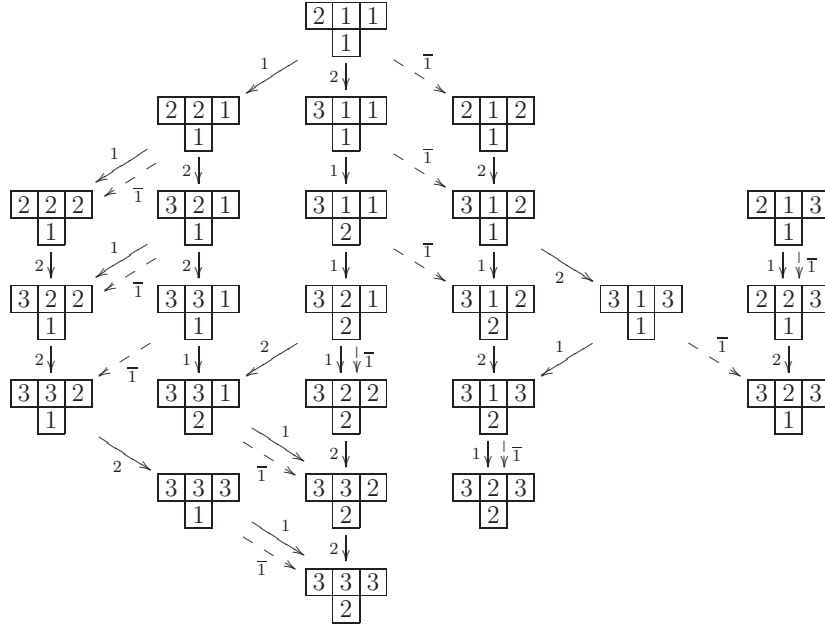


FIGURE 1. $\mathbf{B}(3\epsilon_1 + \epsilon_2)$ for $n = 3$.

2.2. The shifted Littlewood-Richardson rule. We present an explicit combinatorial rule of decomposing the tensor product of crystal bases of $U_q(\mathfrak{q}(n))$ -modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. This algorithm is an analogue of the rule of decomposing the tensor product of crystal bases of $U_q(\mathfrak{gl}(n))$ -modules in [16] (see also [7]), which coincides with the classical *Littlewood-Richardson rule*.

Let λ be a strict partition. We define $\lambda \leftarrow j$ to be the array of cells obtained from the shifted shape λ by adding a cell at the j -th row. Let us denote by $\lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_r$ the array of cells obtained from $\lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_{r-1}$ by adding a cell at the j_r -th row.

We define $\mathbf{B}(\lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_r)$ to be the null crystal (i.e., the empty set) unless $\lambda \leftarrow j_1 \leftarrow \cdots \leftarrow j_k$ is a shifted shape for all $k = 1, \dots, r$.

Theorem 2.8. *Let λ and μ be strict partitions. Then there is a $\mathfrak{q}(n)$ -crystal isomorphism*

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \simeq \bigoplus_{u_1 u_2 \cdots u_N \in \mathbf{B}(\lambda)} \mathbf{B}(\mu \leftarrow (n - u_N + 1) \leftarrow (n - u_{N-1} + 1) \leftarrow \cdots \leftarrow (n - u_1 + 1)),$$

where $N = |\lambda|$.

Proof. Let $u_1 \cdots u_N \in \mathbf{B}(\lambda)$. The vector $u_1 \cdots u_N \otimes L^\mu$ is a lowest weight vector in $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ if and only if $w_0 \mu + \epsilon_{u_N} + \epsilon_{u_{N-1}} + \cdots + \epsilon_{u_k} \in w_0 \Lambda^+$ for all $k = 1, \dots, N$ by Corollary 1.16. This condition is equivalent to

$$\mathbf{B}(\mu \leftarrow (n - u_N + 1) \leftarrow (n - u_{N-1} + 1) \leftarrow \cdots \leftarrow (n - u_1 + 1)) \neq \emptyset.$$

Note that for a lowest weight vector $u_1 \cdots u_N \otimes L^\mu$, we have

$$\begin{aligned} C(u_1 \cdots u_N \otimes L^\mu) &\simeq B(w_0(\text{wt}(u_1 \cdots u_N)) + \mu) \\ &\simeq \mathbf{B}(\mu \leftarrow (n - u_N + 1) \leftarrow (n - u_{N-1} + 1) \leftarrow \cdots \leftarrow (n - u_1 + 1)). \end{aligned}$$

Thus we have the desired result. \square

Therefore, we obtain an explicit description of *shifted Littlewood-Richardson coefficients*.

Corollary 2.9. *Define*

$$\begin{aligned} \mathcal{LR}_{\lambda, \mu}^\nu := \{ &u = u_1 \cdots u_N \in \mathbf{B}(\lambda) ; \quad \text{(a) } \text{wt}(u) = w_0(\nu - \mu) \text{ and} \\ &\text{(b) } \mu + \epsilon_{n-u_N+1} + \cdots + \epsilon_{n-u_k+1} \in \Lambda^+ \text{ for all } 1 \leq k \leq N \}, \end{aligned}$$

and set $f_{\lambda, \mu}^\nu := |\mathcal{LR}_{\lambda, \mu}^\nu|$. Then there is a $\mathfrak{q}(n)$ -crystal isomorphism

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \simeq \bigoplus_{\nu \in \Lambda^+} \mathbf{B}(\nu)^{\oplus f_{\lambda, \mu}^\nu}.$$

Example 2.10. Let $n = 3$, $\lambda = 2\epsilon_1$ and $\mu = 3\epsilon_2 + \epsilon_1$. For $u_1 u_2 \in \mathbf{B}(\lambda)$, if $u_2 = 1$ then

we have $\mu \leftarrow (3 - u_2 + 1) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$ so that $\mathbf{B}(\mu \leftarrow (3 - u_2 + 1) \leftarrow (3 - u_1 + 1)) = \emptyset$.

For the other $u_1 u_2 \in \mathbf{B}(\lambda)$, $\mu \leftarrow (3 - u_2 + 1) \leftarrow (3 - u_1 + 1)$ is given as follows:

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \hline & & \\ \hline & & * \\ \hline & & \bullet \\ \hline \end{array} (u_1 u_2 = 12), & \begin{array}{|c|c|c|c|} \hline & & & * \\ \hline & & & \\ \hline & & & \bullet \\ \hline \end{array} (u_1 u_2 = 13), & \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & * \\ \hline & & & \bullet \\ \hline \end{array} (u_1 u_2 = 22), \\ \begin{array}{|c|c|c|c|} \hline & & & * \\ \hline & & & \bullet \\ \hline & & & \\ \hline \end{array} (u_1 u_2 = 23), & \begin{array}{|c|c|c|c|} \hline & & & \bullet \\ \hline & & & * \\ \hline & & & \\ \hline \end{array} (u_1 u_2 = 32), & \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & * \\ \hline & & & & \bullet \\ \hline \end{array} (u_1 u_2 = 33). \end{array}$$

Here, $\boxed{*}$ and $\boxed{\bullet}$ denotes the cell added at the first and the second step, respectively. Hence we have

$$\mathcal{LR}_{\lambda,\mu}^{3\epsilon_1+2\epsilon_2+\epsilon_3} = \{12\}, \quad \mathcal{LR}_{\lambda,\mu}^{4\epsilon_1+2\epsilon_2} = \{23, 32\}, \quad \mathcal{LR}_{\lambda,\mu}^{5\epsilon_1+\epsilon_2} = \{33\}.$$

It follows that

$$\mathbf{B}(2\epsilon_1) \otimes \mathbf{B}(3\epsilon_1 + \epsilon_2) \simeq \mathbf{B}(3\epsilon_1 + 2\epsilon_2 + \epsilon_3) \oplus \mathbf{B}(4\epsilon_1 + 2\epsilon_2)^{\oplus 2} \oplus \mathbf{B}(5\epsilon_1 + \epsilon_2).$$

3. INSERTION SCHEME

3.1. Knuth relation. Recall that there is an equivalence relation on the set of three letter words, which is called the *Knuth relation*, on $\mathfrak{gl}(n)$ -crystals [1]. In this section we introduce an equivalence relation on the set of four letter words. It is a special case of $\mathfrak{q}(n)$ -crystal equivalence.

Definition 3.1. Let B_i be an abstract $\mathfrak{q}(n)$ -crystals and let $b_i \in B_i$ ($i = 1, 2$). We say that b_1 is $\mathfrak{q}(n)$ -crystal equivalent to b_2 if there exists an isomorphism of crystals

$$B_1 \supseteq C(b_1) \xrightarrow{\sim} C(b_2) \subseteq B_2,$$

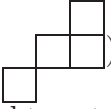
sending b_1 to b_2 . We denote this equivalence relation by $b_1 \sim b_2$.

Example 3.2. By Corollary 1.18, we know that $nnnn$, $(n-1)nnn$ and $n(n-1)nn$ exhaust all the lowest weight vectors in $\mathbf{B}^{\otimes 4}$. Since $\text{wt}((n-1)nnn) = \text{wt}(n(n-1)nn) = w_0(3\epsilon_1 + \epsilon_2)$, we have $C((n-1)nnn) \simeq C(n(n-1)nn) \simeq B(3\epsilon_1 + \epsilon_2)$, and hence $(n-1)nnn \sim n(n-1)nn$. This $\mathfrak{q}(n)$ -crystal equivalence is a special case of the following proposition.

Proposition 3.3 (queer Knuth relation). *Let B_1 and B_2 be the connected components containing 1121 and 1211 in $\mathbf{B}^{\otimes 4}$, respectively. Then there exists an abstract $\mathfrak{q}(n)$ -crystal isomorphism $\psi : B_1 \rightarrow B_2$ such that*

$$\begin{aligned} (3.1) \quad & \psi(abcd) = acbd \quad \text{if } d \leq b \leq a < c \\ (3.2) \quad & \text{or } b < d \leq a < c \\ (3.3) \quad & \text{or } b \leq a < d \leq c \\ (3.4) \quad & \text{or } a < b < d \leq c, \\ (3.5) \quad & = bacd \quad \text{if } b < d \leq c \leq a \\ (3.6) \quad & \text{or } d \leq b < c \leq a, \\ (3.7) \quad & = abdc \quad \text{if } a < d \leq b < c \\ (3.8) \quad & \text{or } d \leq a < b < c. \end{aligned}$$

Proof. Let $\mathbf{B}(Y_1) = \{abcd \in \mathbf{B}^{\otimes 4}; b < c \geq d\}$ and $\mathbf{B}(Y_2) = \{abcd \in \mathbf{B}^{\otimes 4}; a < b \geq c\}$. Then $\mathbf{B}(Y_1)$ (respectively, $\mathbf{B}(Y_2)$) is the set of semistandard tableaux of skew shape

(respectively, of skew shape ) and they are abstract $\mathfrak{q}(n)$ -crystals([4]).

Because 1121 is the only highest weight vector in $\mathbf{B}(Y_1)$, we conclude that $B_1 = \mathbf{B}(Y_1)$. Similarly, $B_2 = \mathbf{B}(Y_2)$. It is straightforward to check that ψ is a bijection between $\mathbf{B}(Y_1)$ and $\mathbf{B}(Y_2)$.

Since B_1 and B_2 are crystal bases for irreducible highest weight $U_q(\mathfrak{q}(n))$ -modules with highest weight $3\epsilon_1 + \epsilon_2$, there exists a unique crystal isomorphism between them. Because the decomposition of $B(3\epsilon_1 + \epsilon_2)$ as a $\mathfrak{gl}(n)$ -crystal is multiplicity free, it is enough to show that ψ is a $\mathfrak{gl}(n)$ -crystal isomorphism between B_1 and B_2 . For a semistandard tableau T and a letter x we denote $T \leftarrow_{\mathfrak{gl}(n)} x$ the tableau obtained by the column insertion x into T . For a word $w = w_1 \cdots w_N$, set $P_{\text{col}}(w) := (\cdots ((w_1 \leftarrow_{\mathfrak{gl}(n)} w_2) \leftarrow_{\mathfrak{gl}(n)} w_3) \cdots) \leftarrow_{\mathfrak{gl}(n)} w_N$. As proved in [1], if $P_{\text{col}}(w) = P_{\text{col}}(w')$ then w is $\mathfrak{gl}(n)$ -crystal equivalent to w' (for the definition of the column insertion scheme and the $\mathfrak{gl}(n)$ -crystal equivalence, see [1]). Thus it is enough to show that $P_{\text{col}}(abcd) = P_{\text{col}}(\psi(abcd))$ for all $abcd \in B_1$. For example, if $d \leq b \leq a < c$, then we have $P_{\text{col}}(abcd) = \begin{array}{|c|c|c|} \hline d & b & a \\ \hline c & & \\ \hline \end{array} = P_{\text{col}}(acbd)$. The other cases can be verified in a similar manner. \square

3.2. Insertion scheme. In this section, we present an algorithm of decomposing the tensor product $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$, using the *insertion scheme* for semistandard decomposition tableaux.

Definition 3.4. (cf. [20]). Let T be a semistandard decomposition tableau of shifted shape λ . For $x \in \mathbf{B}$, we define $T \leftarrow x$ to be a filling of an array of cells obtained from T by applying the following procedure:

- (a) Let $v_1 = u_1 \cdots u_m$ be the reading word of the first row of T such that $u_1 \geq \cdots \geq u_k < \cdots < u_m$ for some $1 \leq k \leq m$. If $v_1 x$ is a hook word, then put x at the end of the first row and stop the procedure.
- (b) Assume that $v_1 x$ is not a hook word. Let u_j be the leftmost element in $v_1 \uparrow$ which is greater than or equal to x . Replace u_j by x . Let u_i be the leftmost element in $v_1 \downarrow$ which is strictly less than u_j . Replace u_i by u_j . (Hence u_i is bumped out of the first row.)
- (c) Apply the same procedure to the second row with u_i as described in (a) and (b).
- (d) Repeat the same procedure row by row from top to bottom until we place a cell at the end of a row of T .

We identify $T \leftarrow x$ with the word which is obtained by reading each row from left to right and then moving to the next row from bottom to top.

Example 3.5. Since

$$\begin{array}{|c|c|c|c|c|} \hline 6 & 6 & 1 & 3 & 5 \\ \hline \end{array} \leftarrow 2 = \begin{array}{|c|c|c|c|c|} \hline 6 & 6 & 3 & 2 & 5 \\ \hline 1 & & & & \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 2 & 4 \\ \hline \end{array} \leftarrow 1 = \begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 3 & & \end{array}$$

we obtain

$$\begin{array}{|c|c|c|c|c|} \hline 6 & 6 & 1 & 3 & 5 \\ \hline 3 & 2 & 4 & & \end{array} \leftarrow 2 = \begin{array}{|c|c|c|c|c|} \hline 6 & 6 & 3 & 2 & 5 \\ \hline 4 & 2 & 1 & & \\ 3 & & & & \end{array}.$$

In the rest of this section, we will show that $T \leftarrow x$ is a semistandard decomposition tableau and it is $\mathfrak{q}(n)$ -crystal equivalent to $T \otimes x$. We need the following lemmas.

Lemma 3.6. *Let $y_1 < x_1 < \cdots < x_N$ for some $N \geq 1$. Then for $z \in \mathbf{B}$, we have*

$$\begin{aligned} (y_1 x_1 \cdots x_N) z &\sim y_1 x_1 \cdots x_N z && \text{if } z > x_N, \\ &\sim y_1 x_i x_1 \cdots x_{i-1} z x_{i+1} \cdots x_N && \text{if } x_{i-1} < z \leq x_i \ (i \geq 2), \\ &\sim y_1 x_1 z x_2 \cdots x_N && \text{if } z \leq x_1. \end{aligned}$$

Proof. If $N = 1$, it is trivial.

Let $N = 2$. Then we have

$$\begin{aligned} (y_1 x_1 x_2) z &\sim y_1 x_1 x_2 z && \text{if } z > x_2, \\ &\sim y_1 x_2 x_1 z && \text{if } x_1 < z \leq x_2 \text{ by (3.4),} \\ &\sim y_1 x_1 z x_2 && \text{if } z \leq x_1 \text{ by (3.7) or (3.8).} \end{aligned}$$

Let $N \geq 3$. If $x_N < z$, it is trivial. For the case $x_{N-1} < z \leq x_N$, we have

$$\begin{aligned} (y_1 x_1 \cdots x_{N-2} x_{N-1} x_N) z &\sim y_1 x_1 \cdots (x_{N-2} x_N x_{N-1} z) && \text{by (3.4)} \\ &\sim y_1 x_1 \cdots (x_{N-3} x_N x_{N-2} x_{N-1}) z && \text{by (3.4)} \\ &\quad \dots \\ &\sim (y_1 x_N x_1 x_2) \cdots x_{N-1} z && \text{by (3.4).} \end{aligned}$$

Let $z \leq x_{N-1}$. Then we have

$$(y_1 x_1 \cdots x_{N-2} x_{N-1} x_N) z \sim y_1 x_1 \cdots (x_{N-2} x_{N-1} z x_N) \quad \text{by (3.7) or (3.8).}$$

Now our assertion follows from induction on N . \square

Lemma 3.7. *Let $y_M \geq y_{M-1} \geq \cdots \geq y_1 < x$ for some $M \geq 1$. Then for $u \in \mathbf{B}$, we have*

$$\begin{aligned} (y_M \cdots y_1 x) u &\sim y_M \cdots y_1 x u && \text{if } u > x, \\ &\sim y_j y_M \cdots y_{j+1} x y_{j-1} \cdots y_1 u && \text{if } u \leq x, \ y_j < x \leq y_{j+1} \ (1 \leq j < M), \\ &\sim y_M x y_{M-1} \cdots y_1 u && \text{if } u \leq x, \ y_M < x. \end{aligned}$$

Proof. If $M = 1$, our assertion is trivial.

Suppose $M \geq 2$. Let $x \geq u$ and $y_j < x$ for some $j \in \{1, 2, \dots, M\}$. Then we have

$$\begin{aligned}
 (y_j \cdots y_2 y_1 x)u &\sim y_j \cdots (y_2 x y_1 u) && \text{by (3.1) or (3.2) or (3.3)} \\
 &\sim y_j \cdots (y_3 x y_2 y_1)u && \text{by (3.1)} \\
 &\quad \dots \\
 &\sim (y_j x y_{j-1} y_{j-2}) \cdots y_1 u && \text{by (3.1).}
 \end{aligned}$$

In particular, we obtain our claim for $j = M$.

If $y_j < x \leq y_{j+1}$ ($1 \leq j < M$), then we have

$$\begin{aligned}
 (y_M \cdots y_j \cdots y_1 x)u &\sim y_M \cdots y_{j+1} (y_j x y_{j-1} y_{j-2}) \cdots y_1 u \\
 &\sim y_M \cdots y_{j+2} (y_j y_{j+1} x y_{j-1}) y_{j-2} \cdots y_1 u && \text{by (3.6)} \\
 &\sim y_M \cdots (y_j y_{j+2} y_{j+1} x) y_{j-2} \cdots y_1 u && \text{by (3.5)} \\
 &\sim y_M \cdots (y_j y_{j+3} y_{j+2} y_{j+1} x) y_{j-1} \cdots y_1 u && \text{by (3.5)} \\
 &\quad \dots \\
 &\sim (y_j y_M y_{M-1} y_{M-2}) \cdots y_{j+1} x y_{j-1} \cdots y_1 u && \text{by (3.5).}
 \end{aligned}$$

□

Lemma 3.8. *Let $\lambda_1 > \lambda_2$. The tensor product $\mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ contains $\mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$ which is the only direct summand isomorphic to $B(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$.*

Proof. If $b_1 \otimes b_2$ is a lowest weight vector in $\mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ of weight $\lambda_2 \epsilon_{n-1} + \lambda_1 \epsilon_n$, then $b_2 = n^{\lambda_1}$ by Lemma 1.15. Since $\text{wt}(b_1) = \lambda_2 \epsilon_{n-1}$, we get $b_1 = (n-1)^{\lambda_2}$. Thus we have $b_1 \otimes b_2 = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2}$ and hence $\mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \subseteq \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$. □

Lemma 3.9. *Let $\lambda_1 > \lambda_2$. We have the following $\mathfrak{q}(n)$ -crystal decomposition.*

$$\mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \otimes \mathbf{B} \simeq B((\lambda_1 + 1)\epsilon_1 + \lambda_2 \epsilon_2) \oplus B(\lambda_1 \epsilon_1 + (\lambda_2 + 1)\epsilon_2) \oplus B(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3),$$

where the second summand appears if and only if $\lambda_1 > \lambda_2 + 1, n \geq 2$ and the third summand appears if and only if $\lambda_2 > 1, n \geq 3$. The corresponding lowest weight vectors are given as follows:

- (a) $(n-1)^{\lambda_2} n^{\lambda_1} \otimes n = L^{(\lambda_1+1)\epsilon_1 + \lambda_2 \epsilon_2},$
- (b) $(n-1)^{\lambda_2} n^{\lambda_1-2} (n-1)n \otimes n$ if $\lambda_1 > \lambda_2 + 1, n \geq 2,$
- (c) $(n-1)^{\lambda_2-2} (n-2)(n-1)n^{\lambda_1-2} (n-1)n \otimes n$ if $\lambda_2 > 1, n \geq 3.$

In particular, we have

$$\mathbf{B}((\lambda_1 + 1)\epsilon_1 + \lambda_2 \epsilon_2) \subseteq \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \otimes \mathbf{B}.$$

Proof. The decomposition follows from $\mathbf{B} \otimes \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \simeq \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \otimes \mathbf{B}$ and Theorem 4.6 (c) in [4].

Note that $(n-1)^{\lambda_2} n^{\lambda_1} \in \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$ and $(n-1)^{\lambda_2} n^{\lambda_1} \otimes n = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2}$. It follows that

$$\mathbf{B}((\lambda_1 + 1)\epsilon_1 + \lambda_2 \epsilon_2) \subseteq \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \otimes \mathbf{B}.$$

Assume $\lambda_1 > \lambda_2 + 1$. The assertion for $\lambda_2 = 0$ and $\lambda_1 = 2$ is trivial. Let $\lambda_2 > 0$. One can easily show that $(n-1)^{\lambda_2} n^{\lambda_1-2} (n-1)n \in \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$, using Proposition 2.3. On the other hand, we have

$$(n-1)^{\lambda_2} n^{\lambda_1-2} (n-1)nn \sim (n-1)^{\lambda_2} n^{\lambda_1-3} (n-1)nnnn \quad \text{by (3.5)}$$

...

$$\sim (n-1)^{\lambda_2+1} n^{\lambda_1} = L^{\lambda_1 \epsilon_1 + (\lambda_2+1)\epsilon_2} \quad \text{by (3.5).}$$

Thus $(n-1)^{\lambda_2} n^{\lambda_1-2} (n-1)n \otimes n$ is a unique lowest weight vector of weight $\lambda_1 \epsilon_1 + (\lambda_2+1)\epsilon_2$.

Assume $\lambda_2 > 1$. One can show that $(n-1)^{\lambda_2-2} (n-2)(n-1)n^{\lambda_1-2} (n-1)n \in \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$ using Proposition 2.3. On the other hand, we have

$$\begin{aligned} & (n-1)^{\lambda_2-2} (n-2)(n-1)n^{\lambda_1-2} (n-1)nn \\ & \sim (n-1)^{\lambda_2-2} (n-2)(n-1)n^{\lambda_1-3} (n-1)nnnn \quad \text{by (3.5)} \end{aligned}$$

...

$$\sim (n-1)^{\lambda_2-2} (n-2)(n-1)^2 n^{\lambda_1} \quad \text{by (3.5)}$$

...

$$\sim (n-2)(n-1)^{\lambda_2} n^{\lambda_1} = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3} \quad \text{by (3.5).}$$

Thus $(n-1)^{\lambda_2-2} (n-2)(n-1)n^{\lambda_1-2} (n-1)n \otimes n$ is a unique lowest weight vector of weight $\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3$. \square

Lemma 3.10. *Let $\lambda_1 > \lambda_2 + 1$. Then $\mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ contains $\mathbf{B}(\lambda_1 \epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ which is the only direct summand isomorphic to $B(\lambda_1 \epsilon_1 + (\lambda_2 + 1)\epsilon_2)$. Moreover, we have*

$$\mathbf{B}(\lambda_1 \epsilon_1 + (\lambda_2 + 1)\epsilon_2) \subseteq \mathbf{B} \otimes \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \subseteq \mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1).$$

Proof. Let $a \otimes b_1 \otimes b_2$ be a lowest weight vector in $\mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ of weight $(\lambda_2 + 1)\epsilon_{n-1} + \lambda_1 \epsilon_n$. Then we have $b_2 = n^{\lambda_1}$ by Lemma 1.15. Comparing the weights, we get $b_1 = (n-1)^{\lambda_2}$ and $a = n-1$. Hence $a \otimes b_1 \otimes b_2 = L^{\lambda_1 \epsilon_1 + (\lambda_2+1)\epsilon_2}$ and $b_1 \otimes b_2 = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2}$. \square

Lemma 3.11. *If $\lambda_1 > \lambda_2 > 1$, then $\mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ contains $\mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3)$ which is the only direct summand isomorphic to $B(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3)$. Moreover, we have*

$$\mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3) \subseteq \mathbf{B} \otimes \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2) \subseteq \mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1).$$

Proof. Let $a \otimes b_1 \otimes b_2$ be a lowest weight vector in $\mathbf{B} \otimes \mathbf{B}(\lambda_2 \epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ of weight $\epsilon_{n-2} + \lambda_2 \epsilon_{n-1} + \lambda_1 \epsilon_n$. Then we have $b_2 = n^{\lambda_1}$ by Lemma 1.15. Hence $a = n-1$ or $n-2$.

If $a = n - 1$, then $b_1 = (n - 1)^{m_1}(n - 2)(n - 1)^{m_2}$ for some nonnegative integers m_1 and m_2 such that $m_1 + m_2 = \lambda_2 - 1$. Since $(n - 2)(n - 1)^{m_2} \otimes n^{\lambda_1}$ is a lowest weight vector by Lemma 1.15, we have $m_2 > 1$ by Corollary 1.18. Then $b_1 = (n - 1)^{m_1}(n - 2)(n - 1)^{m_2}$ is not a hook word, which is a contradiction.

If $a = n - 2$, then $b_1 = (n - 1)^{\lambda_2}$ and $a \otimes b_1 \otimes b_2 = (n - 2)(n - 1)^{\lambda_2} n^{\lambda_1} = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3}$ and $b_1 \otimes b_2 = L^{\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2}$, as desired. \square

Lemma 3.12. *Let $\lambda_1 > \lambda_2 + 1$. Then $\mathbf{B}((\lambda_2 + 1)\epsilon_1) \otimes \mathbf{B}(\lambda_1 \epsilon_1)$ does not have direct summands isomorphic to $B(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2 + \epsilon_3)$.*

Proof. If $\lambda_2 \leq 1$, there is nothing to prove. Let $\lambda_2 > 1$. If $b_1 \otimes b_2$ is a lowest weight vector of weight $\epsilon_{n-2} + \lambda_2 \epsilon_{n-1} + \lambda_1 \epsilon_n$, then $b_2 = n^{\lambda_1}$, by Lemma 1.15. Then $b_1 = (n - 1)^{m_1}(n - 2)(n - 1)^{m_2}$ for some nonnegative integers m_1 and m_2 with $m_1 + m_2 = \lambda_2$. By Lemma 1.15, $(n - 2)(n - 1)^{m_2} \otimes n^{\lambda_1}$ is a lowest weight vector, and hence $m_2 > 1$ by Corollary 1.18. Then $b_1 = (n - 1)^{m_1}(n - 2)(n - 1)^{m_2}$ is not a hook word, which is a contradiction. \square

Now we are ready to prove the main result of this section.

Proposition 3.13. *Let T be a semistandard decomposition tableau of shifted shape λ and let $x \in \mathbf{B}$. Then we have*

- (a) $T \otimes x \sim T \leftarrow x$,
- (b) $T \leftarrow x$ is a semistandard decomposition tableau of shifted shape $\lambda + \epsilon_j$ for some $j = 1, \dots, n$.

Proof. (a) Let v_1 be the reading word of the first row of T . If $v_1 x$ is a hook word, then we have $v_1 \leftarrow x = v_1 \otimes x$ and hence $T \otimes x \sim T \leftarrow x$.

Assume that $v_1 x$ is not a hook word and $v_1 \leftarrow x = y_{j_1} \otimes v'_1$, where v'_1 is the hook word of length λ_1 obtained from v_1 by inserting x into v_1 and y_{j_1} is the letter bumped out of v_1 . Combining Lemma 3.6 and Lemma 3.7, we obtain

$$v_1 \otimes x \sim v_1 \leftarrow x = y_{j_1} \otimes v'_1.$$

Let v_2 be the reading word of the second row of T . If $v_2 y_{j_1}$ is a hook word, then we have

$$v_2(v_1 \otimes x) \sim v_2(v_1 \leftarrow x) = v_2 \otimes (y_{j_1} \otimes v'_1) = (v_2 v_1) \leftarrow x,$$

and hence

$$T \otimes x \sim T \leftarrow x.$$

If $v_2 y_{j_1}$ is not a hook word, then by inserting y_{j_1} into v_2 , we obtain y_{j_2} and v'_2 such that $v_2 \leftarrow y_{j_1} \sim y_{j_2} \otimes v'_2$.

Repeating this procedure row by row, we obtain the desired result.

(b) By the definition of the insertion scheme, it suffices to show that $b_1 \otimes b_2 \leftarrow x$ is a semistandard decomposition tableau for any $x \in \mathbf{B}$ and $b_1 \otimes b_2 \in \mathbf{B}(\lambda_1 \epsilon_1 + \lambda_2 \epsilon_2)$

with $\lambda_1 > \lambda_2$. Note that $\mathbf{B}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2) \subseteq \mathbf{B}(\lambda_2\epsilon_1) \otimes \mathbf{B}(\lambda_1\epsilon_1)$ by Lemma 3.8. It is straightforward to verify our claim for $\lambda_2 = 0$. Let $\lambda_2 > 0$.

If $b_2 \otimes x$ is a hook word, then $b_1 \otimes b_2 \otimes x \in \mathbf{B}((\lambda_1 + 1)\epsilon_1 + \lambda_2\epsilon_2)$. Indeed, if there is a hook subword u of $b_1 \otimes b_2 \otimes x$ of length greater than $\lambda_1 + 1$, then u must contain x . Since $u - \{x\}$ is a hook subword of $b_1 \otimes b_2$ of length greater than λ_1 , we have a contradiction.

Suppose that $b_2 \otimes x$ is not a hook word. We have $b_2 \otimes x \sim y \otimes b'_2$, where b'_2 is the word obtained from b_2 by inserting x into b_2 and y is the element bumped out of b_2 . Note that $b'_2 \in \mathbf{B}(\lambda_1\epsilon_1)$.

Case 1: $b_1 \otimes y$ is not a hook word.

We have $b_1 \otimes y \sim z \otimes b'_1$, where b'_1 is the word obtained from b_1 by inserting y into b_1 and z is the element bumped out of b_1 . It follows that $z \otimes b'_1 \otimes b'_2 \sim b_1 \otimes b_2 \otimes x$. Since $b_2 \otimes x$ is not a hook word, $b_1 \otimes b_2 \otimes x$ does not lie in $\mathbf{B}((\lambda_1 + 1)\epsilon_1 + \lambda_2\epsilon_2)$ and hence it lies in $B(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ or in $B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \epsilon_3)$ in the direct sum decomposition of $\mathbf{B}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2) \otimes \mathbf{B}$, by Lemma 3.9. Since $z \otimes b'_1 \otimes b'_2 \sim b_1 \otimes b_2 \otimes x$, $z \otimes b'_1 \otimes b'_2$ lies in $B(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ or in $B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \epsilon_3)$ in the direct sum decomposition of $\mathbf{B} \otimes \mathbf{B}(\lambda_2\epsilon_1) \otimes \mathbf{B}(\lambda_1\epsilon_1)$. By Lemma 3.10 and Lemma 3.11, in both cases we conclude that $z \otimes b'_1 \otimes b'_2$ is a semistandard decomposition tableau.

Case 2: $b_1 \otimes y$ is a hook word.

Since $b_2 \otimes x$ is not a hook word, we have $b_1 \otimes b_2 \otimes x$ lies in $B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \epsilon_3)$ or in $B(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ in the decomposition of $\mathbf{B}(\lambda_1\epsilon_1 + \lambda_2\epsilon_2) \otimes \mathbf{B}$, by Lemma 3.9. Then, from $b_1 \otimes y \otimes b'_2 \sim b_1 \otimes b_2 \otimes x$, we have $b_1 \otimes y \otimes b'_2 \in B(\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \epsilon_3) \oplus B(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ in the decomposition of $\mathbf{B}((\lambda_2 + 1)\epsilon_1) \otimes \mathbf{B}(\lambda_1\epsilon_1)$. By Lemma 3.12, we get $b_1 \otimes y \otimes b'_2 \in B(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$. Since $\mathbf{B}((\lambda_2 + 1)\epsilon_1) \otimes \mathbf{B}(\lambda_1\epsilon_1) \subseteq \mathbf{B} \otimes \mathbf{B}(\lambda_2\epsilon_1) \otimes \mathbf{B}(\lambda_1\epsilon_1)$, we conclude that $b_1 \otimes y \otimes b'_2 \in \mathbf{B}(\lambda_1\epsilon_1 + (\lambda_2 + 1)\epsilon_2)$ by Lemma 3.10, as desired. \square

Let T and T' be semistandard decomposition tableaux. We define $T \leftarrow T'$ to be

$$(\cdots((T \leftarrow u_1) \leftarrow u_2) \cdots) \leftarrow u_N,$$

where $u_1 u_2 \cdots u_N$ is the reading word of T' .

Corollary 3.14. *Let T and T' be semistandard decomposition tableaux of shifted shape λ and μ , respectively. Then $T \leftarrow T'$ is a semistandard decomposition tableau and we have*

$$T \otimes T' \sim T \leftarrow T'.$$

Proof. Applying Proposition 3.13 (b) repeatedly, we conclude that $T \leftarrow T'$ is a semistandard decomposition tableau. Let $u_1 u_2 \cdots u_N$ be the reading word of T' . Then we have

$$\begin{aligned} T \otimes T' &= (\cdots((T \otimes u_1) \otimes u_2) \cdots) \otimes u_N \\ &\sim (\cdots((T \leftarrow u_1) \otimes u_2) \cdots) \otimes u_N \end{aligned}$$

$$\begin{aligned}
& \sim (\cdots ((T \leftarrow u_1) \leftarrow u_2) \cdots) \otimes u_N \\
& \quad \cdots \\
& \sim (\cdots ((T \leftarrow u_1) \leftarrow u_2) \cdots) \leftarrow u_N. \\
& = T \leftarrow T'.
\end{aligned}$$

□

We now give an algorithm of decomposing the tensor product of $\mathfrak{q}(n)$ -crystals using the insertion scheme.

Theorem 3.15. *We have the following decomposition of tensor product of $\mathfrak{q}(n)$ -crystals.*

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \simeq \bigoplus_{\substack{T \in \mathbf{B}(\lambda) ; \\ T \leftarrow L^\mu = L^\nu \text{ for some } \nu \in \Lambda^+}} \mathbf{B}(\text{sh}(T \leftarrow L^\mu)).$$

Proof. To decompose $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ into a disjoint union of connected $\mathfrak{q}(n)$ -crystals, it is enough to find all the lowest weight vectors. Let $T \otimes T' \in \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ be a lowest weight vector. By Lemma 1.15, we know $T' = L^\mu$. By Corollary 3.14, $T \otimes L^\mu$ is lowest weight vector if and only if $T \leftarrow L^\mu$ is a lowest weight vector, hence we get the desired result. □

Example 3.16. Let $n = 3$, $\lambda = 2\epsilon_1$ and $\mu = 3\epsilon_1 + \epsilon_2$. We have

$$\{T \in \mathbf{B}(2\epsilon_1) ; T \leftarrow L^{3\epsilon_1 + \epsilon_2} \text{ is a lowest weight vector}\} = \{\boxed{1\ 2}, \boxed{2\ 3}, \boxed{3\ 2}, \boxed{3\ 3}\}.$$

Indeed, we obtain

$$\begin{aligned}
\boxed{1\ 2} \leftarrow L^{3\epsilon_1 + \epsilon_2} &= (((\boxed{1\ 2} \leftarrow 2) \leftarrow 3) \leftarrow 3) \leftarrow 3 = \left(\left(\begin{array}{|c|c|} \hline 2 & 2 \\ \hline & 1 \\ \hline \end{array} \leftarrow 3 \right) \leftarrow 3 \right) \leftarrow 3 \\
&= \left(\begin{array}{|c|c|c|} \hline 2 & 2 & 3 \\ \hline & & 1 \\ \hline \end{array} \leftarrow 3 \right) \leftarrow 3 = \begin{array}{|c|c|c|} \hline 3 & 2 & 3 \\ \hline & 1 & 2 \\ \hline \end{array} \leftarrow 3 \\
&= \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline & 2 & 2 \\ \hline & & 1 \\ \hline \end{array},
\end{aligned}$$

and similarly we have

$$\begin{aligned}
\boxed{2\ 3} \leftarrow L^{3\epsilon_1 + \epsilon_2} &= L^{4\epsilon_1 + 2\epsilon_2}, & \boxed{3\ 2} \leftarrow L^{3\epsilon_1 + \epsilon_2} &= L^{4\epsilon_1 + 2\epsilon_2}, \\
\boxed{3\ 3} \leftarrow L^{3\epsilon_1 + \epsilon_2} &= L^{5\epsilon_1 + \epsilon_2}.
\end{aligned}$$

For the other vectors in $\mathbf{B}(2\epsilon_1)$, we have

$$\begin{aligned} \boxed{1|1} \leftarrow L^{3\epsilon_1+\epsilon_2} &= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 3 \\ \hline 1 & 1 & & \\ \hline \end{array}, & \boxed{2|1} \leftarrow L^{3\epsilon_1+\epsilon_2} &= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 3 \\ \hline & 2 & 1 & \\ \hline \end{array}, \\ \boxed{3|1} \leftarrow L^{3\epsilon_1+\epsilon_2} &= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 3 \\ \hline & 1 & 2 & \\ \hline \end{array}, & \boxed{2|2} \leftarrow L^{3\epsilon_1+\epsilon_2} &= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 2 & 3 \\ \hline & 2 & 2 & \\ \hline \end{array}, \\ \boxed{1|3} \leftarrow L^{3\epsilon_1+\epsilon_2} &= \begin{array}{|c|c|c|c|} \hline 3 & 3 & 3 & 3 \\ \hline & 1 & 2 & \\ \hline \end{array}. \end{aligned}$$

Hence we conclude

$$\mathbf{B}(2\epsilon_1) \otimes \mathbf{B}(3\epsilon_1 + \epsilon_2) \simeq \mathbf{B}(3\epsilon_1 + 2\epsilon_2 + \epsilon_1) \oplus \mathbf{B}(4\epsilon_1 + 2\epsilon_2)^{\oplus 2} \oplus \mathbf{B}(5\epsilon_1 + \epsilon_2).$$

4. THE SHIFTED LITTLEWOOD-RICHARDSON TABLEAUX

In this section, we will present two sets of shifted tableaux which parameterize the connected components in the tensor products of $\mathfrak{q}(n)$ -crystals $\mathbf{B}^{\otimes N}$ and $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$, respectively.

Let λ and μ be strict partitions with $\mu \subseteq \lambda$. A filling of the skew shifted shape λ/μ is called a *standard shifted tableau of shape λ/μ* if

- (a) the entries in each row are strictly increasing from left to right,
- (b) the entries in each column are strictly increasing from top to bottom,
- (c) it contains each of the letters $1, 2, \dots, |\lambda/\mu|$ exactly once.

We denote by $\mathcal{ST}(\lambda/\mu)$ the set of standard shifted tableaux of shape λ/μ .

4.1. Decomposition of $\mathbf{B}^{\otimes N}$. There exists a well-known bijection between the set of words with entries $\{1, 2, \dots, n\}$ and the set of pairs (P, Q) , where P is a semistandard Young tableau and Q is a standard Young tableau of the same shape as P . This is called the *Robinson-Schensted-Knuth correspondence*. It can be understood as a decomposition of the $\mathfrak{gl}(n)$ -crystal $\mathbf{B}^{\otimes N}$ into a disjoint union of connected components (see, for example, [9]). Using the insertion scheme presented in the above section, we can get an analogous decomposition of the $\mathfrak{q}(n)$ -crystal $\mathbf{B}^{\otimes N}$.

Definition 4.1. Let $u = u_1 \cdots u_N \in \mathbf{B}^{\otimes N}$.

- (a) The insertion tableau $P(u)$ of u is the semistandard decomposition tableau given by

$$P(u) = (\cdots ((u_1 \leftarrow u_2) \leftarrow u_3) \cdots) \leftarrow u_N.$$

- (b) The recording tableau $Q(u)$ of u is the filling of the shifted shape $\text{sh}(P(u))$ constructed as follows:

- (i) the filling $Q(u)$ consists of the cells that are created by the insertion $(\cdots ((u_1 \leftarrow u_2) \leftarrow u_3) \cdots) \leftarrow u_N$,
- (ii) if u_i is inserted into $(\cdots ((u_1 \leftarrow u_2) \leftarrow u_3) \cdots) \leftarrow u_{i-1}$ to create a cell at the position c_i , then we fill the cell at c_i with the entry i .

Note that for any $u \in \mathbf{B}^{\otimes N}$, $Q(u)$ is a standard shifted tableau with the same shape as $P(u)$.

Example 4.2. (a) Let $n = 3$ and $u = 2321$. Since

$$((2 \leftarrow 3) \leftarrow 2) \leftarrow 1 = (\boxed{2 \ 3} \leftarrow 2) \leftarrow 1 = \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 2 & \\ \hline \end{array} \leftarrow 1 = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 2 & & \\ \hline \end{array},$$

we have

$$P(u) = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 2 & & \\ \hline \end{array}, \quad Q(u) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array}.$$

(b) Let $n = 4$, $\lambda = (6, 4, 2, 1)$, and $u = 12233334444444$. Then we have

$$P(u) = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 3 & 3 & 3 & & \\ \hline 2 & 2 & & & & \\ \hline 1 & & & & & \\ \hline \end{array}, \quad Q(u) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 7 & 8 & 13 \\ \hline 3 & 5 & 9 & 12 & & \\ \hline 6 & 10 & & & & \\ \hline 11 & & & & & \\ \hline \end{array}.$$

Thus we get a map

$$\Psi : \mathbf{B}^{\otimes N} \rightarrow \bigsqcup_{\substack{\lambda \in \Lambda^+ \\ \text{with } |\lambda|=N}} \mathbf{B}(\lambda) \times \mathcal{ST}(\lambda)$$

given by

$$u = u_1 \cdots u_N \mapsto (P(u), Q(u)).$$

The inverse algorithm Ψ^{-1} of Ψ is given as follows: For $P \in \mathbf{B}(\lambda)$ and $Q \in \mathcal{ST}(\lambda)$, let Q_k be the standard shifted tableau obtained from Q by removing the cells with entries $k+1, k+2, \dots, N$ and let x_k be the letter in P at the cell in the same position as $Q_k - Q_{k-1}$ for each k .

- (a) If x_N lies in the first row of P , then set $u_N := x_N$.
- (b) Suppose that x_N lies in the ℓ -th row of P ($\ell \geq 2$). Let $v = y_1 \cdots y_{\lambda_{\ell-1}}$ be the reading word of $(\ell-1)$ -th row of P . Suppose that

$$y_1 \geq \cdots \geq y_k < \cdots < y_{\lambda_{\ell-1}}.$$

If $k = 1$ or $x_N \geq y_1$, then $x_N v$ is a hook word of length $\lambda_{\ell-1} + 1$. Hence we have $k > 1$ and $x_N < y_1$. Let y_i be the rightmost element in $y_1 \cdots y_{k-1}$ which is strictly greater than x_N . Replace y_i by x_N . Let y_j be the rightmost element in $y_k \cdots y_{\lambda_{\ell-1}}$ which is less than or equal to y_i . Replace y_j by y_i . (Hence y_j gets bumped out of v .)

- (c) Apply the same procedure to the $(\ell-2)$ -th row of P with y_j as described in (a) and (b).
- (d) Repeat the same procedure row by row from bottom to top until an element, say u_N , gets bumped out of the first row.

- (e) Let P_{N-1} be the filling of the array of shape $\text{sh}(Q_{N-1})$ obtained from P by applying (a), (b), (c), and (d). Repeat (a), (b), (c), and (d) with T_{N-1} and Q_{N-1} so that an element, say u_{N-1} , gets bumped out of P_{N-1} .
- (f) Repeat the whole procedure until we get N -many letters u_N, \dots, u_1 to get a word $u = u_1 \cdots u_N \in \mathbf{B}^{\otimes N}$.

As a consequence, Ψ is a bijection between $\mathbf{B}^{\otimes N}$ and $\bigsqcup_{\substack{\lambda \in \Lambda^+ \\ \text{with } |\lambda|=N}} \mathbf{B}(\lambda) \times \mathcal{ST}(\lambda)$.

Lemma 4.3. *Let $T \in \mathbf{B}(\lambda)$ and $T' \in \mathbf{B}(\mu)$ for some strict partitions λ and μ . If $T \sim T'$, then $\lambda = \mu$ and $T = T'$.*

Proof. Let $T^\lambda = \tilde{e}_{i_1}^{a_1} \tilde{f}_{i_1}^{b_1} \cdots \tilde{e}_{i_r}^{a_r} \tilde{f}_{i_r}^{b_r} T$ for some $i_1, \dots, i_r \in \{1, \dots, n-1, \bar{1}\}$, and $a_1, \dots, a_r, b_1, \dots, b_r \in \mathbf{Z}_{\geq 0}$. Since $T \sim T'$, $\tilde{e}_{i_1}^{a_1} \tilde{f}_{i_1}^{b_1} \cdots \tilde{e}_{i_r}^{a_r} \tilde{f}_{i_r}^{b_r} T'$ is a highest weight vector in $\mathbf{B}(\mu)$. It must be T^μ , since $\mathbf{B}(\mu)$ has a unique highest weight vector. Because $\text{wt}(T) = \text{wt}(T')$, we get $\lambda = \mu$. It follows that $T^\lambda = T^\mu$ and hence $T = T'$. \square

Lemma 4.4. *Let $u = u_1 \cdots u_N \in \mathbf{B}^{\otimes N}$ and let $i \in \{1, \dots, n-1, \bar{1}\}$. If $\tilde{f}_i u \neq 0$, then $Q(\tilde{f}_i u) = Q(u)$ and if $\tilde{e}_i u \neq 0$, then $Q(\tilde{e}_i u) = Q(u)$.*

Proof. Let $\tilde{f}_i u = u'_1 \cdots u'_N$ such that $u'_j = u_j$ for $j \neq k$ and $i+1 = u'_k = \tilde{f}_i u_k$ for some $1 \leq k \leq N$. It is enough to show that $\text{sh}((\cdots((u_1 \leftarrow u_2) \cdots) \leftarrow u_t)) = \text{sh}((\cdots((u'_1 \leftarrow u'_2) \cdots) \leftarrow u'_t))$ for all $1 \leq t \leq N$.

If $1 \leq t < k$, our assertion is trivial.

If $t \geq k$, then

$$\begin{aligned} (\cdots((u'_1 \leftarrow u'_2) \cdots) \leftarrow u'_t) &\sim u'_1 \otimes \cdots \otimes u'_t \\ &= \tilde{f}_i(u_1 \otimes \cdots \otimes u_t) \quad \text{by Lemma 1.10} \\ &\sim \tilde{f}_i(\cdots((u_1 \leftarrow u_2) \cdots) \leftarrow u_t). \end{aligned}$$

By Lemma 4.3, we have

$$(\cdots((u'_1 \leftarrow u'_2) \cdots) \leftarrow u'_t) = \tilde{f}_i(\cdots((u_1 \leftarrow u_2) \cdots) \leftarrow u_t).$$

Hence, by Theorem 2.5, we have

$$\begin{aligned} \text{sh}((\cdots((u'_1 \leftarrow u'_2) \cdots) \leftarrow u'_t)) &= \text{sh}(\tilde{f}_i(\cdots((u_1 \leftarrow u_2) \cdots) \leftarrow u_t)) \\ &= \text{sh}((\cdots((u_1 \leftarrow u_2) \cdots) \leftarrow u_t)), \end{aligned}$$

as desired.

The proof for $\tilde{e}_i u$ is similar. \square

For a standard shifted tableau Q with $|\text{sh}(Q)| = N$, we define

$$\mathbf{B}_Q = \{u_1 \cdots u_N \in \mathbf{B}^{\otimes N}; Q(u) = Q\}.$$

Now we prove one of the main results of this section.

Theorem 4.5. *We have the following decomposition of the $\mathfrak{q}(n)$ -crystal $\mathbf{B}^{\otimes N}$ into a disjoint union of connected components:*

$$\mathbf{B}^{\otimes N} = \bigoplus_{\substack{\lambda \in \Lambda^+ \\ \text{with } |\lambda|=N}} \left(\bigoplus_{Q \in \mathcal{ST}(\lambda)} \mathbf{B}_Q \right),$$

where \mathbf{B}_Q is isomorphic to $\mathbf{B}(\lambda)$ with $sh(Q) = \lambda$.

Proof. As a set, we have

$$\mathbf{B}^{\otimes N} = \bigsqcup_{\substack{\lambda \in \Lambda^+ \\ \text{with } |\lambda|=N}} \left(\bigsqcup_{Q \in \mathcal{ST}(\lambda)} \mathbf{B}_Q \right).$$

Let Q be a standard shifted tableau of shape $\lambda \in \Lambda^+$. By Lemma 4.4, $\mathbf{B}_Q \cup \{0\}$ is closed under the Kashiwara operators. Since Ψ is a bijection, we have

$$\mathbf{B}_Q = \{\Psi^{-1}(T, Q) ; T \in \mathbf{B}(\lambda)\}.$$

It follows that the map $P : \mathbf{B}_Q \rightarrow \mathbf{B}(\lambda)$ given by $u \mapsto P(u)$ is a bijection.

For any word $u \in \mathbf{B}^{\otimes N}$ and $i \in \{1, \dots, n-1, \bar{1}\}$, we know

$$P(\tilde{f}_i u) \sim \tilde{f}_i u \sim \tilde{f}_i P(u).$$

By Lemma 4.3, we get $P(\tilde{f}_i u) = \tilde{f}_i P(u)$. Similarly we have $P(\tilde{e}_i u) = \tilde{e}_i P(u)$. Hence $P : \mathbf{B}_Q \rightarrow \mathbf{B}(\lambda)$ is a $\mathfrak{q}(n)$ -crystal isomorphism. \square

As an immediate consequence, we obtain the following corollary.

Corollary 4.6. *For a strict partition λ with $|\lambda| = N$, let f^λ be the number of standard shifted tableaux of shape λ . Then we have*

$$\mathbf{B}^{\otimes N} \simeq \bigoplus_{\substack{\lambda \in \Lambda^+ \\ \text{with } |\lambda|=N}} \mathbf{B}(\lambda)^{\oplus f^\lambda}.$$

4.2. The shifted Littlewood-Richardson tableaux. In this section, we will define the notion of *shifted Littlewood-Richardson tableaux of skew shape*, which parameterize the connected components of the tensor product $\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$.

For semistandard decomposition tableaux T and T' , we define $T \rightarrow T'$ by

$$u_1 \leftarrow (u_2 \leftarrow \cdots \leftarrow (u_{N-1} \leftarrow (u_N \leftarrow T') \cdots)),$$

where $u_1 u_2 \cdots u_N$ is the reading word of T .

Definition 4.7. *Let $T \in \mathbf{B}(\lambda)$ and $T' \in \mathbf{B}(\mu)$ for some strict partitions λ, μ and let $sh(T \rightarrow T') = \nu$. The recording tableau $Q(T \rightarrow T')$ of the insertion $T \rightarrow T'$ is the filling of the skew shifted shape ν/μ constructed as follows:*

- (i) the filling $Q(T \rightarrow T')$ consists of the cells that are created by the insertion $T \rightarrow T'$,
 (ii) if $u_{N-k+2} \leftarrow (\cdots \leftarrow (u_{N-1} \leftarrow (u_N \leftarrow T')) \cdots)$ is inserted into u_{N-k+1} to create a cell at the position c_k , then we fill the cell at c_k with the entry k .

Example 4.8. (a) Let $T = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $T' = \begin{bmatrix} 3 & 3 & 3 \\ 2 \end{bmatrix}$. Then we have

$$T \rightarrow T' = \begin{bmatrix} 1 \end{bmatrix} \leftarrow \left(\begin{bmatrix} 2 \end{bmatrix} \leftarrow \begin{bmatrix} 3 & 3 & 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \end{bmatrix} \leftarrow \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 2 & 2 \\ 1 \end{bmatrix}.$$

Hence the recording tableau $Q(T \rightarrow T')$ is given by

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & & 2 \\ \hline \end{array}.$$

(b) Let $T = \begin{bmatrix} 3 & 1 & 2 \\ 2 \end{bmatrix}$ and $T' = \begin{bmatrix} 3 & 2 & 2 \\ 1 \end{bmatrix}$. Then we have

$$T \rightarrow T' = \begin{bmatrix} 3 & 3 & 2 & 2 \\ 2 & 2 & 1 \\ 1 \end{bmatrix}, \quad Q(T \rightarrow T') = \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & 4 \\ \hline & & 2 & \\ \hline \end{array}.$$

Remark 4.9. (a) For any insertion $T \rightarrow T'$, the recording tableau $Q(T \rightarrow T')$ is a standard shifted tableau of shape ν/μ , where $\text{sh}(T') = \mu$ and $\text{sh}(T \rightarrow T') = \nu$.

(b) Since

$$\begin{aligned} & u_1 \leftarrow (u_2 \leftarrow \cdots \leftarrow (u_{N-1} \leftarrow (u_N \leftarrow T')) \cdots) \\ & \sim u_1 \otimes (u_2 \leftarrow \cdots \leftarrow (u_{N-1} \leftarrow (u_N \leftarrow T')) \cdots) \\ & \quad \cdots \\ & \sim u_1 \otimes (u_2 \otimes \cdots \otimes (u_{N-1} \otimes (u_N \leftarrow T')) \cdots) \\ & \sim u_1 \otimes (u_2 \otimes \cdots \otimes (u_{N-1} \otimes (u_N \otimes T')) \cdots), \end{aligned}$$

we have $T \rightarrow T' \sim T \otimes T' \sim T \leftarrow T'$, and $T \rightarrow T' = T \leftarrow T'$ by Lemma 4.3.

Lemma 4.10. Let $T \in \mathbf{B}(\lambda)$ and $T' \in \mathbf{B}(\mu)$ for some strict partitions λ, μ . If $T \rightarrow L^\mu = L^\nu$ for some strict partition ν , then the reading word of T is given by

$$(4.1) \quad (n - r_{|\lambda|} + 1) \otimes (n - r_{|\lambda|-1} + 1) \otimes \cdots \otimes (n - r_1 + 1),$$

where r_k denotes the row of the entry k in $Q(T \rightarrow T')$.

Proof. Let $|\lambda| = N$ and let $u_1 \cdots u_N$ be the reading word of T . Since

$$L^\nu = T \rightarrow L^\mu \sim u_1 \otimes \cdots \otimes u_N \otimes L^\mu,$$

we see that $u_k \otimes \cdots \otimes u_N \otimes L^\mu$ is a lowest weight vector for all $k = 1, 2, \dots, N$, by Corollary 1.16. Then we have

$$u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_{N-1} \leftarrow (u_N \leftarrow L^\mu)) \cdots) = L^{\mu + \epsilon_{n-u_N+1} + \cdots + \epsilon_{n-u_k+1}}$$

for all $k = 1, 2, \dots, N$, by Lemma 4.3. Hence the cell created by inserting $(u_{k+1} \leftarrow \cdots (u_N \leftarrow L^\mu) \cdots)$ into u_k lies at the $(n - u_k + 1)$ -th row of $L^{\mu + \epsilon_{n-u_N+1} + \cdots + \epsilon_{n-u_k+1}}$. Thus we have $u_k = n - r_{N-k+1} + 1$ for all $k = 1, 2, \dots, N$, as desired. \square

For a standard shifted tableau Q of shape ν/μ , let

$$\mathbf{B}(\lambda, \mu)_Q = \{T \otimes T' \in \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) ; Q(T \rightarrow T') = Q\}.$$

Proposition 4.11. *For strict partitions λ, μ, ν with $\mu \subseteq \nu$, let Q be a standard shifted tableau of shape ν/μ .*

- (a) *The set $\mathbf{B}(\lambda, \mu)_Q \cup \{0\}$ is closed under the action of the Kashiwara operators.*
- (b) *The $\mathfrak{q}(n)$ -crystal $\mathbf{B}(\lambda, \mu)_Q$ is isomorphic to $\mathbf{B}(\nu)$.*

Proof. (a) Let $T \in \mathbf{B}(\lambda)$, $T' \in \mathbf{B}(\mu)$ and let $u_1 \cdots u_N$ be the reading word of T . Fix an $i \in \{1, 2, \dots, n-1, \bar{1}\}$. We will only show that $\mathbf{B}(\lambda, \mu)_Q \cup \{0\}$ is closed under \tilde{f}_i since the proof for \tilde{e}_i is similar.

Suppose that $\tilde{f}_i(T \otimes T') = \tilde{f}_i T \otimes T'$. Then there exists $k \in \{1, 2, \dots, N\}$ such that

$$\tilde{f}_i(T \otimes T') = u_1 \otimes \cdots \otimes \tilde{f}_i u_k \otimes \cdots \otimes u_N \otimes T'.$$

Observe that for all $j \leq k$,

$$\begin{aligned} & u_j \leftarrow (u_{j+1} \leftarrow \cdots (\tilde{f}_i u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots) \\ & \sim u_j \otimes u_{j+1} \otimes \cdots \otimes \tilde{f}_i u_k \otimes u_{k+1} \otimes \cdots \otimes u_N \otimes T' \\ & = \tilde{f}_i(u_j \otimes u_{j+1} \otimes \cdots \otimes u_k \otimes u_{k+1} \otimes \cdots \otimes u_N \otimes T') \quad \text{by Lemma 1.10} \\ & \sim \tilde{f}_i(u_j \leftarrow (u_{j+1} \leftarrow \cdots (u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots)). \end{aligned}$$

Thus, by Lemma 4.3, we have

$$\begin{aligned} & \tilde{f}_i(u_j \leftarrow (u_{j+1} \leftarrow \cdots (u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots)) \\ & = u_j \leftarrow (u_{j+1} \leftarrow \cdots (\tilde{f}_i u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots). \end{aligned}$$

It follows that

$$\begin{aligned} & \text{sh}(u_j \leftarrow (u_{j+1} \leftarrow \cdots (u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots)) \\ & = \text{sh}(u_j \leftarrow (u_{j+1} \leftarrow \cdots (\tilde{f}_i u_k \leftarrow (u_{k+1} \leftarrow \cdots (u_N \leftarrow T') \cdots)) \cdots)) \end{aligned}$$

for all $j \leq k$. Hence we get $Q(T \rightarrow T') = Q(\tilde{f}_i T \rightarrow T')$.

By a similar argument, one can show that if $\tilde{f}_i(T \otimes T') = T \otimes \tilde{f}_i T'$ then $Q(T \rightarrow T') = Q(T \rightarrow \tilde{f}_i T')$.

(b) If $T \otimes T'$ is a lowest weight vector in the $\mathfrak{q}(n)$ -crystal $\mathbf{B}(\lambda, \mu)_Q$, then we have $T' = L^\mu$ and $T \rightarrow T' = L^\nu$. By Lemma 4.10, we get

$$T = (n - r_{|\lambda|} + 1) \otimes (n - r_{|\lambda|-1} + 1) \otimes \cdots \otimes (n - r_1 + 1),$$

where r_k denotes the row of the entry k in $Q(T \rightarrow T')$. So the $\mathfrak{q}(n)$ -crystal $\mathbf{B}(\lambda, \mu)_Q$ in $\mathbf{B}^{\otimes(|\lambda|+|\mu|)}$ has a unique lowest weight vector $T \otimes T'$ which is $\mathfrak{q}(n)$ -crystal equivalent to L^ν . It follows that $\mathbf{B}(\lambda, \mu)_Q \simeq \mathbf{B}(\nu)$, as desired. \square

Now we define the notion of *shifted Littlewood-Richardson tableaux*.

Definition 4.12. Let λ, μ, ν be strict partitions with $\mu \subseteq \nu$. We define

$$\widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu := \{Q \in \mathcal{ST}(\nu/\mu); (n - r_{|\lambda|} + 1) \otimes \cdots \otimes (n - r_1 + 1) \in \mathbf{B}(\lambda), \\ \text{where } r_k \text{ denotes the row of the entry } k \text{ in } Q\}.$$

A tableau Q in $\widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu$ is called a shifted Littlewood-Richardson tableau of shape ν/μ and type λ .

Let $T_1 \otimes T_2 \in \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ and $\text{sh}(T_1 \rightarrow T_2) = \nu$ for some strict partition ν . Since $T_1 \otimes T_2 \sim T_1 \rightarrow T_2$, there exists a $\mathfrak{q}(n)$ -crystal isomorphism

$$\psi : C(T_1 \otimes T_2) \rightarrow \mathbf{B}(\nu)$$

sending $T_1 \otimes T_2$ to $T_1 \rightarrow T_2$. Let $T \otimes L^\mu = \psi^{-1}(L^\nu)$ for some $T \in \mathbf{B}(\lambda)$. We obtain $T \rightarrow L^\mu = L^\nu$. By Proposition 4.11 (a), we have $Q(T_1 \rightarrow T_2) = Q(T \rightarrow L^\mu)$. Note that $Q(T \rightarrow L^\mu) \in \widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu$, by Lemma 4.10. Hence we have $Q(T_1 \rightarrow T_2) \in \widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu$.

Conversely, if $Q \in \widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu$, then there exists an element $T_1 \otimes T_2 \in \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ such that $Q(T_1 \rightarrow T_2) = Q$. For example, if $T_1 = (n - r_{|\lambda|} + 1) \otimes (n - r_{|\lambda|-1} + 1) \otimes \cdots \otimes (n - r_1 + 1)$ and $T_2 = L^\mu$, then we have $Q(T_1 \rightarrow T_2) = Q$, where r_k denotes the row of the entry k in Q .

To summarize, we obtain another main result of this section.

Theorem 4.13. Let λ, μ be strict partitions. Then there exists a decomposition of the tensor product of $\mathfrak{q}(n)$ -crystals as follows:

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) = \bigoplus_{\substack{\nu \in \Lambda^+ \\ \text{with } \mu \subseteq \nu}} \bigoplus_{Q \in \widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu} \mathbf{B}(\lambda, \mu)_Q.$$

Consequently, we have $|\widetilde{\mathcal{LR}}_{\lambda, \mu}^\nu| = f_{\lambda, \mu}^\nu$.

Example 4.14. (a) For $n = 3$, let $\lambda = 2\epsilon_1$, $\mu = 3\epsilon_1 + \epsilon_2$, $\nu_1 = 3\epsilon_1 + 2\epsilon_2 + \epsilon_3$, $\nu_2 = 4\epsilon_1 + 2\epsilon_2$ and $\nu_3 = 5\epsilon_1 + \epsilon_2$. For each $i = 1, 2, 3$ the set $\widetilde{\mathcal{LR}}_{\lambda, \mu}^{\nu_i}$ is the same

as the set $\mathcal{ST}(\nu_i/\mu)$, since any word of length 2 is a semistandard decomposition tableau of shifted shape λ . So we have

$$\begin{aligned}\widetilde{\mathcal{LR}}_{\lambda,\mu}^{\nu_1} &= \left\{ \begin{array}{|c|c|c|} \hline & & \\ \hline & & 1 \\ \hline & & 2 \\ \hline \end{array} \right\}, \\ \widetilde{\mathcal{LR}}_{\lambda,\mu}^{\nu_2} &= \left\{ \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & \\ \hline & & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & & 1 \\ \hline & & \\ \hline \end{array} \right\}, \\ \widetilde{\mathcal{LR}}_{\lambda,\mu}^{\nu_3} &= \left\{ \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & 2 \\ \hline & & & \\ \hline \end{array} \right\}.\end{aligned}$$

Hence we obtain the same decomposition as in Example 2.10, and Example 3.16.

- (b) For $n = 3$, let $\lambda = \mu = 3\epsilon_1 + \epsilon_2$ and $\nu = 4\epsilon_1 + 3\epsilon_2 + \epsilon_3$. Then we have

$$\mathcal{ST}(\nu/\mu) = \left\{ \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & & 4 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 4 \\ \hline & & 3 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & & 1 & 3 \\ \hline & & 4 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 2 \\ \hline & & 1 & 4 \\ \hline & & 3 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & 4 \\ \hline & & 2 & \\ \hline \end{array} \right\}.$$

One can easily check that

$$\widetilde{\mathcal{LR}}_{\lambda,\mu}^{\nu} = \left\{ \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 2 & 3 \\ \hline & & 4 & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline & & & 3 \\ \hline & & 1 & 4 \\ \hline & & 2 & \\ \hline \end{array} \right\},$$

and hence $f_{\lambda,\mu}^{\nu} = 2$. From similar calculations, we obtain the following decomposition of tensor product of $\mathfrak{q}(n)$ -crystals:

$$\begin{aligned}\mathbf{B}(3\epsilon_1 + \epsilon_2) \otimes \mathbf{B}(3\epsilon_1 + \epsilon_2) &\simeq \mathbf{B}(6\epsilon_1 + 2\epsilon_2) \oplus \mathbf{B}(5\epsilon_1 + 3\epsilon_2)^{\oplus 2} \\ &\quad \oplus \mathbf{B}(5\epsilon_1 + 2\epsilon_2 + \epsilon_3)^{\oplus 2} \oplus \mathbf{B}(4\epsilon_1 + 3\epsilon_2 + \epsilon_3)^{\oplus 2}.\end{aligned}$$

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